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*ON COWLING'S THEOREM ON THE IMPOSSIBILITY OF SELF-MAINTAINED AXISYMMETRIC HOMOGENEOUS DYNAMOS**

By G. E. Backus and S. Chandrasekhar

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Communicated January 31, 1956

1. In current attempts to explain the maintenance of cosmic magnetic fields in terms of currents induced in moving conducting fluid by an already existing field in the manner of a self-exciting dynamo, the following theorem due to Cowling[1] has played an important part: *When the magnetic field and the fluid motions are symmetric about an axis and the lines of force of the magnetic field as well as the trajectories of the fluid particles are confined to meridional planes, no stationary dynamo can exist.* As Bullard[2] has expressed, "Many people believe that this theorem is a foretaste of stronger theorems that will prove the process impossible." Whether or not this is the case, a stronger theorem is to the effect that no dynamo is possible even when we do not restrict the magnetic lines of force and the trajectories of the fluid particles to meridional planes, and allow toroidal fields and rotational motions as well—still, however, in the axisymmetric framework of constant density and constant conductivity. It is possible that this extension of Cowling's theorem is known to those who have worked in this field (cf. Elsasser[3]); but we are not aware that a formal proof has been published. In this paper we shall present such a proof. It provides a further illustration of the methods used in two recent communications.[4]

2. The basic equations of the dynamo problem are

$$\boldsymbol{E} = \frac{1}{4\pi\sigma} \operatorname{curl} \boldsymbol{H} - \boldsymbol{v} \times \boldsymbol{H} \tag{1}$$

and

$$\operatorname{curl} \boldsymbol{E} = -\frac{\partial \boldsymbol{H}}{\partial t} = 0, \tag{2}$$

where σ denotes the electrical conductivity of the medium and the remaining symbols have their usual meanings. The dynamo problem, quite generally, is therefore one of finding the conditions under which $\boldsymbol{E}$ given by equation (1) can be expressed as the gradient of a scalar function.

We shall express $\boldsymbol{H}$ in the form (cf. Chandrasekhar,[4] eq. [3])

$$\boldsymbol{H} = -\varpi \frac{\partial P}{\partial z}\mathbf{1}_{\varpi} + \varpi T\mathbf{1}_{\varphi} + \frac{1}{\varpi}\frac{\partial}{\partial \varpi}(\varpi^2 P)\mathbf{1}_z, \tag{3}$$

where P and T are two scalar functions. Since $\rho\boldsymbol{v}$ is also a solenoidal vector and is assumed to be axisymmetric, we can similarly express it in terms of two scalar functions U and V, in the manner

$$\rho\boldsymbol{v} = -\varpi \frac{\partial U}{\partial z}\mathbf{1}_{\varpi} + \varpi V\mathbf{1}_{\varphi} + \frac{1}{\varpi}\frac{\partial}{\partial \varpi}(\varpi^2 U)\mathbf{1}_z. \tag{4}$$

With the foregoing expressions for $\boldsymbol{H}$ and $\boldsymbol{v}$, we find

$$\begin{aligned}\boldsymbol{E} = &\left\{-\frac{\varpi}{4\pi\sigma}\frac{\partial T}{\partial z} - \frac{V}{\rho}\frac{\partial}{\partial \varpi}(\varpi^2 P) + \frac{T}{\rho}\frac{\partial}{\partial \varpi}(\varpi^2 U)\right\}\mathbf{1}_{\varpi} \\ &+ \left\{-\frac{\varpi}{4\pi\sigma}\Delta_5 P + \frac{1}{\rho}\frac{\partial P}{\partial z}\frac{\partial}{\partial \varpi}(\varpi^2 U) - \frac{1}{\rho}\frac{\partial U}{\partial z}\frac{\partial}{\partial \varpi}(\varpi^2 P)\right\}\mathbf{1}_{\varphi} \\ &+ \left\{\frac{1}{4\pi\sigma\varpi}\frac{\partial}{\partial \varpi}(\varpi^2 T) + \frac{\varpi^2 T}{\rho}\frac{\partial U}{\partial z} - \frac{\varpi^2 V}{\rho}\frac{\partial P}{\partial z}\right\}\mathbf{1}_z,\end{aligned} \tag{5}$$

where it may be recalled that Δ_5 is the Laplacian operator for axisymmetric functions in five-dimensional Euclidean space. We require that

$$\text{curl } \boldsymbol{E} = -\frac{\partial E_{\varphi}}{\partial z}\mathbf{1}_{\varpi} + \left(\frac{\partial E_{\varpi}}{\partial z} - \frac{\partial E_z}{\partial \varpi}\right)\mathbf{1}_{\varphi} + \frac{1}{\varpi}\frac{\partial}{\partial \varpi}(\varpi E_{\varphi})\mathbf{1}_z \tag{6}$$

vanish identically.

It is apparent from equation (6) that the vanishing of the ϖ- and the z-components of curl $\boldsymbol{E}$ requires that

$$E_{\varphi} \equiv 0. \tag{7}$$

Hence

$$\frac{\varpi}{4\pi\sigma}\Delta_5 P = \frac{1}{\rho}\frac{\partial P}{\partial z}\frac{\partial}{\partial \varpi}(\varpi^2 U) - \frac{1}{\rho}\frac{\partial U}{\partial z}\frac{\partial}{\partial \varpi}(\varpi^2 P). \tag{8}$$

We can write this alternatively in the form

$$\varpi^3\Delta_5 P = \frac{4\pi\sigma}{\rho}\frac{\partial(\varpi^2 P, \varpi^2 U)}{\partial(z, \varpi)}. \tag{9}$$

The vanishing of the φ-component of curl $\boldsymbol{E}$ requires that

$$\frac{\partial E_{\varpi}}{\partial z} - \frac{\partial E_z}{\partial \varpi} = 0. \tag{10}$$

Using the expressions for E_{ϖ} and E_z given in equation (5), we find that this condition reduces to

$$\frac{1}{4\pi}\frac{\partial}{\partial z}\left(\frac{\varpi}{\sigma}\frac{\partial T}{\partial z}\right) + \frac{1}{4\pi}\frac{\partial}{\partial \varpi}\left(\frac{1}{\sigma\varpi}\frac{\partial}{\partial \varpi}\varpi^2 T\right) = \frac{\partial(T/\rho, \varpi^2 U)}{\partial(z, \varpi)} - \frac{\partial(V/\rho, \varpi^2 P)}{\partial(z, \varpi)}. \tag{11}$$

3. It can be verified that equation (9) is equivalent to the one which Cowling has derived;[1] and Cowling proves that the boundary conditions on P are such that

$$P \equiv 0. \tag{12}$$

He argues as follows: The total magnetic induction across an area perpendicular to Oz bounded by a circle of radius ϖ with center on Oz is equal to $2\pi\varpi^2 P$. Accordingly, $\varpi^2 P = \psi$ (say) must vanish on Oz and at infinity. Now, in terms of ψ, equation (9) reduces to

$$\frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial^2\varpi} - \frac{1}{\varpi}\frac{\partial\psi}{\partial\varpi} = \frac{4\pi\sigma}{\rho}\frac{\partial(\psi, \varpi^2 U)}{\partial(z, \varpi)}. \tag{13}$$

Since ψ vanishes on $\varpi = 0$ and at infinity, it must have an extremum at some point in the (ϖ, z)-plane. At this point grad $\psi = 0$, while div (grad ψ) $\neq 0$; this contradicts equation (13). This argument is not strict, since it is possible that div (grad ψ) $= 0$ at the extremum: the extremum might be of a higher order. But this objection to Cowling's theorem can be disposed of by appealing to the following theorem on elliptic equations.[5]

Suppose that a_{ij}, b_j, and c are real functions in an n-dimensional Euclidean space. Suppose, further, that in the interior of a volume V in this space they are continuous, $c \leq 0$, and the matrix a_{ij} is positive definite. Suppose also that in the interior of V,

$$\sum_{i,j} a_{ij}\frac{\partial^2\psi}{\partial x_i\,\partial x_j} + \sum_j b_j\frac{\partial\psi}{\partial x_j} + c\psi = 0 \tag{14}$$

and that on the surface of V, ψ vanishes. Then ψ vanishes throughout V.

Cowling's proof that equation (13) and the boundary conditions on ψ imply $\psi \equiv 0$ is essentially equivalent to the proof of the foregoing theorem in the case $c < 0$. The case $c = 0$ has to be treated separately, but with some additional care the theorem is true in this case also; this is shown by Courant and Hilbert.[5]

4. Turning now to equation (11), we have (on setting $P = 0$)

$$\frac{\partial}{\partial z}\left(\frac{\varpi}{\sigma}\frac{\partial T}{\partial z}\right) + \frac{\partial}{\partial\varpi}\left(\frac{1}{\sigma\varpi}\frac{\partial}{\partial\varpi}\varpi^2 T\right) = 4\pi\frac{\partial(T/\rho, \varpi^2 U)}{\partial(z, \varpi)}. \tag{15}$$

We shall discuss this equation only when ρ and σ are constants, in which case it reduces to

$$\varpi\Delta_5 T = \frac{4\pi\sigma}{\rho}\frac{\partial(T, \varpi^2 U)}{\partial(z, \varpi)}. \tag{16}$$

Since $-\varpi\partial U/\rho\partial z$ and $\partial(\varpi^2 U)/\rho\varpi\partial\varpi$ are the ϖ- and the z-components of the velocity, we can rewrite equation (15) in the form

$$\Delta_5 T = 4\pi\sigma\boldsymbol{v}\cdot\text{grad } T. \tag{17}$$

The boundary condition on T is that it vanish on the bounding surface. For concreteness, suppose that the surface of the dynamo is a sphere of radius R. We have already remarked that Δ_5 is the axisymmetric Laplacian in five-dimensional space and that the boundary-value problem we wish to solve is genuinely one in five-

dimensions. That we can appeal to the theorem on elliptic equations quoted in Section 3 can be seen as follows:

Define

$$f(x_1, x_2, x_3, x_4, z) = T(\sqrt{(x_1^2 + x_2^2 + x_3^2 + x_4^2)}, z). \tag{18}$$

Then

$$\Delta_5 T = \sum_{i=1}^{4} \frac{\partial^2 f}{\partial x_i^2} + \frac{\partial^2 f}{\partial z^2}, \tag{19}$$

and equation (17) can be rewritten in the form

$$\sum_{i=1}^{4} \frac{\partial^2 f}{\partial x_i^2} + \frac{\partial^2 f}{\partial z^2} - 4\pi\sigma\left(v_\varpi \frac{\partial T}{\partial \varpi} + v_z \frac{\partial T}{\partial z}\right) = 0. \tag{20}$$

Since $\boldsymbol{v}$ is axisymmetric, and if it is everywhere differentiable in the ordinary three-dimensional space, $v_\varpi = \varpi h(\varpi, z)$, where h is continuous. But

$$\varpi \frac{\partial T}{\partial \varpi} = \sum_{i=1}^{4} x_i \frac{\partial f}{\partial x_i}; \tag{21}$$

and if we define

$$v_i(x_1, x_2, x_3, x_4) = x_i h(\sqrt{(x_1^2 + x_2^2 + x_3^2 + x_4^2)}, z) \quad (i = 1, \ldots, 4), \tag{22}$$

then the v_i's are continuous, and equation (20) becomes

$$\sum_{i=1}^{4} \frac{\partial^2 f}{\partial x_i^2} + \frac{\partial^2 f}{\partial z^2} - 4\pi\sigma\left(\sum_{i=1}^{4} v_i \frac{\partial f}{\partial x_i} + v_z \frac{\partial f}{\partial z}\right) = 0. \tag{23}$$

Here f vanishes on the surface of the five-dimensional sphere $x_1^2 + x_2^2 + x_3^2 + x_4^2 + z^2 = R^2$ and satisfies equation (23) everywhere in the interior of that sphere, except possibly at the points $\varpi = 0$. All that remains necessary, in order to show that the theorem of Section 3 can be applied to equation (23) to make $f = 0$, is to show that equation (22) holds even at $\varpi = 0$ and that f is twice continuously differentiable there.

It is not difficult to show from equation (3) that if $\boldsymbol{H}$ is three times continuously differentiable in Cartesian co-ordinates, then $T(\varpi, z)$ is twice continuously differentiable in the closed half-plane $\varpi \geq 0$ of the (ϖ, z)-plane and satisfies $\partial T/\partial \varpi = 0$ at $\varpi = 0$. But these facts are enough to establish that f defined by equation (18) is twice continuously differentiable everywhere in the five-dimensional space. Since equation (23) is satisfied everywhere in $x_1^2 + x_2^2 + x_3^2 + x_4^2 + z^2 \leq R^2$, except possibly when $x_1 = x_2 = x_3 = x_4 = 0$, and all the terms of equation (23) are continuous, equation (23) must be satisfied even when $x_1 = x_2 = x_3 = x_4 = 0$. Consequently $f \equiv 0$ in the interior of the sphere.

We conclude, therefore, that no steady-state axisymmetric dynamo with constant conductivity and density can exist if the Cartesian components of its magnetic field are three times continuously differentiable.

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EQUATIONS TO OBTAIN, FOR EQUILIBRIUM REACTIONS, FREE-ENERGY, HEAT, AND ENTROPY CHANGES FROM TWO CALORIMETRIC MEASUREMENTS

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Among the classical methods for the experimental determination of driving forces of processes of life or nonliving matter, ΔF, ΔH, and ΔS (the free-energy, heat, and entropy changes), two main approaches may be distinguished: (*a*) Material quantities, namely, mole fractions of reactants and products at equilibrium, have been measured at different temperatures to obtain ΔF and ΔH by chemical analysis. (*b*) Quantities of energy, namely, heats of combustion or reaction, and heat capacities at various temperatures, extrapolated to absolute zero, have been measured by calorimetry to obtain ΔH and ΔS.

In a symposium at San Francisco in 1935, Rossini[1] introduced as the ultimate end of chemical thermodynamics the compilation of the master-table of data, notably ΔF. In the same symposium, Eastman[2] remarked on the chemical and thermal ways of measurement as follows: "In 1906 Nernst published his famous 'heat theorem' as the consummation of the long cherished ambition of the chemist to determine affinities from purely thermal data."

It will be shown in the present paper that the definition of chemical equilibrium and its relation to the standard free-energy change, ΔF°, also permit, in a most elementary fashion, another way to determine affinity, ΔF°, and simultaneously the standard entropy change, ΔS°, from purely thermal data, namely, from two experimental quantities, Q_1 and Q_2 [cal/mole], heats of reaction. For an equilibrium reaction,

$$A + B \rightleftharpoons C + D,$$

the first experimental quantity, Q_1 [cal/mole], is obtained by mixing very dilute solutions of pure reactants, $A + B$ (and, if necessary, a suitable catalyst or enzyme), in a microcalorimeter, where the total quantity of heat absorbed between the beginning and the end of heat production is recorded. The second experimental quantity, Q_2 [cal/mole], is obtained by going through the same procedure at the same temperature with very dilute solutions of pure products, $C + D$. Further, we make the assumption that with the very low concentrations of the species under

study, partial molar heat contents are constant and differences between molarities and activities are negligible. Then the experimental quantities Q_1 and Q_2 are related to ΔH and ΔF° of the reaction under study by two simple equations,

$$\Delta H = Q_1 - Q_2 \tag{1}$$

and

$$\Delta F^\circ = -RT \ln \left(\frac{Q_1}{-Q_2} \right)^2. \tag{2}$$

With a new method of microcalorimetry capable of measuring reaction heats with solutions of 10^{-3}–10^{-5} molarity,[3, 4] which has in fact stimulated the present study and will be described in detail in a separate publication, equations (1) and (2) offer several striking advantages.

Compared with the classical *chemical* approach, one quantitative method and instrument of analysis, namely, microcalorimetry, has been substituted—with respect to this particular objective—for any conceivable number of chemical analytical procedures, each specially adapted to only one group of reactions or even to only one reaction. The simplicity thereby achieved may stimulate development or refinement of techniques for other linear properties[5] associated with all processes, for example, volume changes.

Moreover, whereas chemical analysis for ΔF° may in some instances disturb the equilibrium state and make its observation impossible, the calorimetric measurement will not disturb the balance. It is completed at the moment when the state of equilibrium has been attained.

Compared with the classical approach from purely *thermal* data, the method here described requires only two experimental observations for the determination of ΔF°, ΔH, and ΔS°. Instead of quantities of the order of 10 gm., as used for third-law entropy determinations, it requires only micromoles of substances, which is advantageous in biochemical studies. No measurements at low temperatures or extrapolations to absolute zero are necessary

One general advantage of the method is the possibility of carrying out an independent check without introducing an independent method of analysis. This may be done by measuring the paired heats, Q_1 and Q_2, twice, at different temperatures, to obtain ΔF° of the reaction at these two temperatures. ΔH, derived from the two values of ΔF° and the temperatures in the Gibbs-Helmholtz equation, must not differ from $\Delta H = Q_1 - Q_2$.

On the other side, the following prerequisites of our method are listed: Solutions must be dilute, $M/100$ or less. The reaction under study must proceed to the equilibrium point within a few hours in vitro. It must not be accompanied by other heat-absorbing processes of a similar order of magnitude. It must have a measurable ΔH and a reasonable equilibrium constant, usually between 10^3 and 10^{-3}, unless it is suited for coupling with another reaction of opposite ΔF° sign and known thermodynamic data. This will, however, quite often be feasible in the case of metabolic reactions.

Equations (1) and (2) can be generalized for equilibrium reactions involving any number of molecules of any number of reactants and products, as follows: For a reaction

$$aA + bB + cC + \ldots \rightleftarrows rR + sS + tT + \ldots$$

treated in the same manner to obtain heat quantities Q_1 and Q_2 [cal/mole], the heat of reaction is

$$\Delta H = Q_1 - Q_2,$$

and the standard free-energy change is

$$\Delta F^\circ = -RT \ln \frac{(Q_1/(Q_1 - Q_2))^{r+s+t+\cdots}}{(-Q_2/(Q_1 - Q_2))^{a+b+c+\cdots}} \frac{[R]_0^r [S]_0^s [T]_0 \ldots}{[A]_0^a [B]_0^b [C]_0^c \ldots}, \quad (3)$$

where $[R]_0[S]_0 \ldots$ are the initial concentrations of products R, S, . . . in Sample I, and $[A]_0[B]_0 \ldots$ are the initial concentrations of reactants A, B, . . . in Sample II.

One useful instance of the general case is of course the asymmetrical reaction

$$A + B \rightleftarrows AB,$$

for which the standard free-energy change is

$$\Delta F^\circ = -RT \ln \frac{(Q_1/(Q_1 - Q_2))}{(-Q_2/(Q_1 - Q_2))^2}. \quad (4)$$

In showing how these equations have been derived on the basis of simple and well-known formulas of thermodynamics, we shall confine ourselves to the simple case of a "symmetrical" reaction of two components on either side. Special cases in which the heat measurements would be done with Samples I and II, initially containing known quantities of both reactants and products, will also be omitted, although such combinations may have experimental advantages in some cases, and their equation was derived in this study earlier than the simpler case given here with unit quantities of

$$A + B \rightleftarrows C + D.$$

Let Sample I initially contain 1 mole of reactants, $A + B$, and let Sample II initially contain 1 mole of products, $C + D$, in very dilute solutions. Then, after the reactions have attained a balance between backward and forward rates, which is indicated by zero rate of energetic output from the microcalorimeter in both experiments, reactants and products will be present in Samples I and II at equilibrium in equal quantities, namely, $1 - x$ moles of reactants and x moles of products, if x is the unknown quantity transformed in Sample I from reactants into products. In Sample II, which initially contained unit quantity of products and no reactants, $1 - x$ moles of products must have been transformed into reactants to attain the equilibrium state. Thus, while in Sample I the transformation of x moles was associated with the total heat change Q_1, the transformation of $1 - x$ moles in the opposite direction in Sample II was associated with the total heat change Q_2.

We may therefore write, for a sufficiently dilute solution of reactants and products,

$$\frac{x}{1 - x} = \frac{Q_1}{-Q_2}$$

or

$$x = \frac{Q_1}{(Q_1 - Q_2)}.$$

At this point, ΔH follows from x, the quantity transformed, and the total heat observed, Q_1, with Sample I:

$$\Delta H = \frac{Q_1}{x}$$

or

$$\Delta H = Q_1 - Q_2. \tag{1}$$

In order to obtain the standard free-energy change, ΔF°, we form, from the quantities—or rather concentrations—in the equilibrium state, the equilibrium constant

$$K = \frac{[C] \times [D]}{[A] \times [B]}$$

or, substituting terms described above,

$$K = \frac{x^2}{(1 - x)^2}$$

or

$$K = \frac{(Q_1/(Q_1 - Q_2))^2}{(1 - Q_1/(Q_1 - Q_2))^2} = \left(\frac{Q_1}{-Q_2}\right)^2,$$

and the standard free-energy change is

$$\Delta F^\circ = -RT \ln \left(\frac{Q_1}{-Q_2}\right)^2. \tag{2}$$

The practical feasibility and precision of the method theoretically outlined here will depend mainly on the quantitative purity of reactants, the magnitudes of ΔH and ΔF°, the rate at which the reactions approach the equilibrium point, and the precision of the thermal measurement. It is, of course, essential that the catalyst or enzyme emerge unchanged from the reaction.

The determination of ΔH and ΔF° by this method will, of course, be favored by high ΔH, because more heat is available for the measurements. On the other hand, conditions are more favorable with low absolute values of ΔF°. With increasing absolute value of ΔF°, the equilibrium constant, K, moves farther from unity. If ΔF° is large and negative, the absorbed heat Q_1 approaches but never exceeds ΔH, while the absorbed heat Q_2 approaches but never attains zero. (The roles of Q_1 and Q_2 are, of course, reversed if ΔF° is large and positive.) It is decisive for the range of usefulness of our procedure that the heat Q_2 could be measured with a similar percentage of accuracy, although it may be smaller than Q_1 by orders of magnitude.[4]

The standard entropy change, ΔS°, follows, of course, from ΔF°, ΔH, and the temperature, T, in the fundamental equation

$$\Delta F = \Delta H - T\Delta S.$$

In summary, it has been shown how chemical affinity, $\Delta F°$, and also the standard entropy change, $\Delta S°$, of equilibrium processes (at low concentrations) may be determined in a most elementary manner from purely thermal data, two heats of reaction, without chemical analysis.

It is my pleasant obligation to thank Dr. Terrell L. Hill for calling my attention to the restriction to very dilute solutions and for the treatment of the general case, equation (3), and Drs. Dean Burk, John Z. Hearon, Sterling B. Hendricks, Hans A. Krebs, and Frederick D. Rossini for their most valuable comments and suggestions.

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THE ARRANGEMENT OF ATOMS IN CRYSTALS OF THE WOLLASTONITE GROUP OF METASILICATES

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Introduction.—A structure for the monoclinic $CaSiO_3$ (variously called "para-wollastonite" or "β wollastonite") was proposed by Barnick.[1] Ito[2] adapted this structure to wollastonite (sometimes called "α wollastonite" to distinguish it from "β wollastonite"), which is the common, triclinic, mineral form of $CaSiO_3$. In these proposed structures, there occurs the same (Si_3O_9) ring which is found in benitoite, $CaTi(SiO_3)_3$.

It is difficult to reconcile a benitoite ring structure with the extremely fibrous nature of the wollastonite cleavage, which points, rather, to an (SiO_3) chain structure of some sort. Furthermore, the comparison of observed and computed intensities does not recommend the structure of wollastonite proposed by Ito. For this reason, the structures of the minerals of the wollastonite group were re-investigated, using the powerful image-seeking function method. An outline of this study and some of its results are presented here. The study establishes that the structures of these minerals are indeed based upon (SiO_3) chains. The structure-determination methods are described in detail in papers which appear elsewhere.

Cells and Symmetries.—Pectolite, $Ca_2NaHSi_3O_9$, and wollastonite, $CaSiO_3$, had been considered monoclinic until Warren and Biscoe[3] incidentally examined them

as possible members of the monoclinic pyroxenes. They found that pectolite and wollastonite, while obviously related to each other, were not closely related to the monoclinic pyroxenes, and indeed were triclinic. Peacock[4] confirmed their triclinic symmetry in a later morphological study.

Bustamite, $CaMn(SiO_3)_2$, was early thought to be a member of the rhodonite group.[5] After a limited X-ray investigation, Berman and Gonyer[6] assigned it to the wollastonite group and called it a "manganese wollastonite."

New cell data for these minerals, obtained from precession and deJong-Bouman photographs, are offered in Table 1. The cell data for pectolite and wollastonite are in good agreement with those of Warren and Biscoe.[3] The data for bustamite are in good agreement with those of Berman and Gonyer,[6] except that the new investigation shows that the cell edges a and c are double those of the wollastonite-type cell and that the space group is $F\bar{1}$ or $F1$, in wollastonite orientation, instead of $P\bar{1}$. Bustamite evidently bears a kind of superstructure relation to wollastonite, and it may be regarded as a double salt, $CaSiO_3 \cdot MnSiO_3$.

TABLE 1

SYMMETRY AND CELL DATA FOR CRYSTALS OF THE WOLLASTONITE GROUP

	Pectolite, $NaHCa_2Si_3O_9$	Wollastonite, $CaSiO_3$	Bustamite, $CaMn(SiO_3)_2$
Space group	$P\bar{1}$	$P\bar{1}$	$F\bar{1}$
a	7.99 Å	7.94 Å	2×7.73 Å
b	7.04 Å	7.32 Å	7.18 Å
c	7.02 Å	7.07 Å	2×6.92 Å
α	90°03′	90°02′	89°34′
β	95°17′	95°22′	94°53′
γ	102°28′	103°26′	102°47′
Cell content	2 $Ca_2NaHSi_3O_9$	6 $CaSiO_3$	12 $CaMn(SiO_3)_2$

Structure-Determination Method.—In a series of papers the writer[7] has shown that a vector set can be solved for its fundamental set of points and that, in principle at least, a Patterson synthesis can be solved approximately for the electron density upon which it is based. To solve a Patterson synthesis, it is necessary to have a start on the solution. The start usually amounts to identification in the Patterson map of a nonmultiple peak, ordinarily a "rotation" peak in a projection. Provided that the individual peaks of the solution are reasonably greater than the Patterson background, a solution is possible.

The structures of pectolite and wollastonite were determined by solving in this manner the Patterson projections parallel to the cell edges. For this purpose the Patterson projections $P(xy)$, $P(yz)$, and $P(xz)$ were prepared for pectolite and wollastonite.

The Patterson map $P(xz)$ for pectolite was readily solved. On it was found a collection of eight peaks which bore the correct geometrical and weight relations of two sets of "rotation" peaks and their two sets of equal, unsymmetrical conjugates. For each of the two distinct rotation peaks as image point, a map of the minimum function $M_2(xz)$ was formed, with a weight proportional to the particular rotation peak. These two maps were very similar and could readily be combined to form the map of the more powerful minimum function $M_4(xz)$. This function bears such a powerful inequality relation to the electron density $\rho(xz)$ that the general nature of the structure could be deduced from it.

It became evident that pectolite and wollastonite have a pseudo-identity period of $b/2$ along the b-axis. This made it possible to regard the crystal as composed of a substructure and a complement structure,[8] so that the complement structure could be studied separately. To this end, not only were the Patterson syntheses of the full projections $P(xy)$ and $P(yz)$ studied but also the Patterson syntheses $\partial P(xy)$ and $\partial P(yz)$ of the complement structure. The latter were transformed to $\partial M_2(xy)$ and $\partial M_2(yz)$, which, together with $M_2(xy)$ and $M_2(yz)$ of the full structure, and with $\rho(xz)$ already determined, permitted the derivation of the additional electron-density projections $\rho(xy)$ and $\rho(yz)$.

The structure of wollastonite was determined independently in an analogous manner. The wollastonite-type structure contains fifteen atoms in the asymmetric unit. This appears to be the largest asymmetric unit solved by direct solution of Patterson projections.

The Structure Type.—The co-ordinates of the atoms in the pectolite structure found as a result of the study just outlined are listed in Table 2. These co-ordinates are still subject to refinement, especially for the oxygen atoms.

TABLE 2

CO-ORDINATES OF ATOMS IN PECTOLITE

	Pectolite		
	x	y	z
Ca_1	0.141	0.406	0.854
Ca_2	.155	.914	.862
Na	.448	.736	.659
H	.177	.615	.539
Si_1	.224	.403	.333
Si_2	.212	.948	.339
Si_3	.450	.743	.145
O_1	.349	.202	.871
O_2	.336	.704	.946
O_3	.184	.490	.542
O_4	.174	.839	.533
O_5	.074	.406	.181
O_6	.048	.880	.198
O_7	.392	.529	.270
O_8	.402	.915	.280
O_9	0.248	0.186	0.383

The wollastonite-type structure contains SiO_3 chains running parallel to the b-axes of the crystals. The shape of the chains is different from that in the pyroxenes. It can be described as a sequence of Si_2O_7 tetrahedron pairs, roughly symmetrical with respect to a plane between the tetrahedra, connected by a third kind of tetrahedron to form a single chain.

The calcium-oxygen arrangement in the wollastonite-type structure is interesting and, in part, unorthodox. To a first, rough approximation, the calcium atoms are octahedrally surrounded by oxygen atoms, and the octahedra share three edges, to form a sheet parallel to (101). In detail, there are three sets of calcium atoms: Ca_1, Ca_2, and Ca_3. The Ca_1 and Ca_2 atoms have truly octohedral oxygen environments and share edges to form a lathlike strip parallel to the b-axis in the sheet parallel to (101). On the other hand, the Ca_3 atom (Na in pectolite) is close to only three of the six oxygen atoms which were crudely described as constituting its octahedral co-ordination, and is much farther from the three other oxygen atoms. This inequality in distance causes rifts in the (101) sheet, so that it is dissected into

a set of parallel lathlike strips which extend indefinitely in the direction of the b-axis and which have their common planar surfaces parallel to (101).

The co-ordination of the Ca_3 (or Na in pectolite) is curious. Its nearest neighbors are the three oxygen atoms O_2, O_3, and O_4. All other oxygen atoms are at distances much greater than the sum of the ionic radii of Ca (or Na) and O. The Ca_3 (or Na) atoms thus have trigonal pyramidal co-ordination, and this pyramid shares edges with the edges of the Ca_1 and Ca_2 octahedra.

The lengths of the a- and c-axes of pectolite and wollastonite are quite close, but pectolite has a considerably shorter b-axis than wollastonite. In the structure this is reflected in the close approach of the O_3 and O_4 atoms of neighboring silicon tetrahedra. Evidently a hydrogen bond occurs between these atoms which results in a relative contraction of the b-axis of pectolite.

More detailed accounts of the determination of the structures of pectolite and wollastonite and more detailed descriptions of these structures are published elsewhere.

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*ALGEBRAIC VARIETIES WITH RATIONAL DISSECTIONS**

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Let V be a variety,[1] and let $\{W_i\}$ be a finite set of varieties contained in V (which need not be closed in V and hence are not, in general, subvarieties in V) such that every point in V is contained in one and only one of these varieties; we shall call such a set of varieties $\{W_i\}$ a *dissection* of V, and the dissection is said to be *rational* if each variety W_i is equivalent to an affine space by an everywhere biregular birational transformation. A dissection $\{W_i\}$ of a variety V is said to be *defined* over a field K if K is a field of definition for all the varieties W_i (and hence necessarily also for V); and in case $\{W_i\}$ is a rational dissection, we add the further condition that

each W_i is equivalent to an affine space by an everywhere biregular birational transformation which is defined over K. It is evident that if W is an element in a (rational) dissection $\{W_i\}$ of V, then the subset consisting of all W_i contained in the closure $\bar{W}$ or the boundary $\bar{W} - W$ of W in V is also a (rational) dissection of $\bar{W}$ or $\bar{W} - W$, respectively. We shall be concerned in this note only with rational dissections, and we shall call an element W in a rational dissection a *cell*, or an *r-cell* if r is the dimension of W. It is clear that a variety having a rational dissection must be a rational variety; in fact, any dissection of a variety V of dimension d can contain only one element W of dimension d, and it is easily seen that V must coincide with the closure $\bar{W}$ of W in V. The converse of this statement is, however, not true in general; for this, one need only observe that if a variety V has a rational dissection, then (in case the universal domain is the complex field) all the odd-dimensional Betti numbers of V must vanish, and that there exist rational varieties with nonvanishing odd-dimensional Betti numbers.

It has been shown by Ehresmann[2] that certain well-known homogeneous varieties, such as the Grassmann varieties, have rational dissections which can be constructed in a simple geometrical way; using topological methods, Ehresmann was able to derive from these dissections many of the fundamental properties of these varieties, in case the universal domain is the complex field. In particular, it follows from Ehresmann's proofs that if a variety has a rational dissection, then the quotient group $G(V, s)/G_n(V, s)$ (where $G(V, s)$ is the group of all s-cycles in V and $G_n(V, s)$ is the subgroup in $G(V, s)$ consisting of all elements which are numerically equivalent to zero) has a finite base for every dimension s; in other words, the basis theorem for numerical equivalence follows from the existence of a rational dissection. It is natural to ask whether this is also true in abstract algebraic geometry and whether we can make the same assertion also for algebraical or rational equivalence; the purpose of this note is to answer these questions in the affirmative. In the case of the Grassmann varieties, this has been done by Hodge and Pedoe[3] by using an idea previously introduced by van der Waerden in his proof of the Bezout theorem.[4] Our method, though not unrelated to this idea of van der Waerden, is much simpler than that of Hodge and Pedoe and is closely analogous to the topological treatment of Ehresmann; and, since we make use only of the existence of a rational dissection, not of its specific form, our results are valid for any variety with such a dissection.

Before proceeding any further, we add a remark concerning the equivalence between cycles in an algebraic variety. The usual equivalence concept used in the basis theorem is the algebraic equivalence, and this is indeed the proper one for the general form of this theorem; however, for our special case it is useful and important to introduce a more strict concept of equivalence, namely, the rational equivalence.[5] Let V be a variety, Z be a cycle in V, and K be a field of definition for V such that Z is rational over K; we shall say that Z is *rationally equivalent to zero over the field* K if there exists a cycle Z_u in V, rational over a purely transcendental extension $K(u)$ of K, such that both Z and the zero cycle are specializations of Z_u over K; and we shall say that a cycle is *rationally equivalent to zero* if it is so over some such field K. Since specialization is compatible with the addition (and subtraction) of cycles, it is easily seen that the set of all cycles of a given dimension s in V which are rationally equivalent to zero is a subgroup $G_r(V, s)$ in the group

$G(V, s)$ of all s-cycles in V; in fact, $G_r(V, s)$ is evidently a subgroup in the group $G_a(V, s)$ of all s-cycles in V which are algebraically equivalent to zero. Two cycles Z_1 and Z_2 in V are said to be *rationally equivalent* to each other if the cycle $Z_1 - Z_2$ is rationally equivalent to zero; and a finite set of s-cycles in V is said to form a *rational base* for the s-cycles in V if every s-cycle in V is rationally equivalent to a linear combination of the cycles in this set. It is clear that in general a variety does not have a rational base for cycles of any dimension; the purpose of this note is to show that in the special case of a variety with rational dissection such a rational base does exist for every dimension.

As we shall see presently, our problem reduces essentially to the proof of the following lemma, which is the algebraical counterpart of the topological lemma used by Ehresmann:

LEMMA. *Let W be an r-cell in a rational dissection of a variety V, and let $B = \bar{W} - W$ be the boundary of W; then any s-cycle Z $(s < r)$ in V with support in $\bar{W}$ is rationally equivalent on $\bar{W}$ to an s-cycle Z_0 whose support is contained in B.*

Remark: The *support* of a cycle Z is the bunch of all varieties which appear as components, with nonvanishing coefficients, in Z.

Proof: It is clearly sufficient to prove the theorem for the case where Z is a prime s-cycle, that is, a variety of dimension s, the general case following obviously by linearity; furthermore, we can evidently assume that the support of Z is not already contained in B, so that the restriction of Z in W is a (nonzero) prime s-cycle X in W, and we have the relation $Z = \bar{X}$. Let F be an everywhere biregular birational transformation of W onto an affine space A^r; let p be a point in W which is not contained in the support of X, and let $x_1, \ldots, x_r$ be a system of affine co-ordinates in A^r, with the origin at the point $F(p)$. Let v be a variable over a field of definition K for $\{W_i\}$ and Z, and let T_v be the linear transformation of A^r defined by the equations $x_i' = x_i/v$; then $F^{-1}T_vF$ is a biregular birational transformation of W onto itself, defined over $K(v)$, so that $F^{-1}T_vF(X) = X_v$ is a prime s-cycle in W, rational over $K(v)$, and hence $Z_v = \bar{X}_v$ is also a prime s-cycle in $\bar{W}$, rational over $K(v)$. Since v is a generic point over K of the affine line A^1, there exists a uniquely determined specialization Z_0 of Z_v over the specialization $v \to 0$ over K; and since Z is evidently the specialization of $\bar{X}_v$ over the specialization $v \to 1$ over K, it follows that Z_0 is rationally equivalent to Z. It remains to show that the support of Z_0 is contained in B. To do this, we imbed A^r in a projective space P^r; then F can be extended to a birational transformation $\bar{F}$ of $\bar{W}$ into P^r, such that the total image of every point in $\bar{W} - W$ is contained in $P^r - A^r$, and conversely. Let $y_0, y_1, \ldots, y_r$ be a system of homogeneous co-ordinates in P^r such that $x_i = y_i/y_0$, $i = 1, \ldots, r$, and let $f(u^{(0)}, u^{(1)}, \ldots, u^{(s)})$ be the associated form of the prime s-cycle $Y = \bar{F}(\bar{X})$ in P^r (the closure of $F(X)$ in P^r), expressed in terms of the system of homogeneous co-ordinates $y_0, y_1, \ldots, y_r$ in P^r; then the form $f_v(u^{(0)}, u^{(1)}, \ldots, u^{(s)})$, obtained from the form f by substituting $vu_0^{(i)}$ for $u_0^{(i)}$, $i = 0, 1, \ldots, s$, is the associated form of the prime s-cycle $Y_v = T_v\bar{F}(\bar{X})$, rational over $K(v)$, whereby T_v is now considered as a projective transformation of P^r onto itself. Over the specialization $v \to 0$ over K, the form f_v specializes to form f_0, which is obtained from f by setting $u_0^{(i)} = 0$, $i = 0, 1, \ldots, s$, and, since the support of the s-cycle Y does not contain the origin in A^r, one sees readily that the form f_0 does not vanish identically;

it follows that f_0 is the associated form of a positive s-cycle in P^r, rational over K, which must be the uniquely determined specialization Y_0 of Y_v over the specialization $v \to 0$ over K. Since the form f_0 does not involve the variables $u_0^{(i)}$, $i = 0, 1, \ldots, s$, any set of s hyperplanes together with the hyperplane $P^r - A^r$, defined by the equation $y_0 = 0$, will have a common intersection with the support of Y_0; this shows that the support of Y_0 is contained in $P^r - A^r$. Now let q be a generic point of $Z_v = \bar{X}_v$ over $K(v)$, which is, of course, also a generic point of X_v over $K(v)$, and consider the prime s-cycle G_v in the product variety $\bar{W} \times P^r$ which is determined by the locus of the point $(q, F(q))$ over $K(v)$; since q and $F(q)$ are generic points of Z_v and Y_v over $K(v)$, respectively, it follows that Z_v and Y_v are the projections of G_v into $\bar{W}$ and P^r, respectively. Furthermore, since $(q, F(q))$ is evidently a point in the graph of F, the support of G_v is contained in the graph of $\bar{F}$, as the latter is the closure of the graph of F in $\bar{W} \times P^r$. If G_0 is the specialization of G_v over the specialization $v \to 0$ over K, then the projections of G_0 into $\bar{W}$ and P^r must be the cycles Z_0 and Y_0, respectively; since the support of G_0 is contained in the graph of $\bar{F}$ and since the support of Y_0 is contained in $P^r - A^r$, it follows that the support of Z_0 is contained in B. This concludes the proof of the lemma.

Consider, now, a rational dissection of a variety V, and let $W_1, \ldots, W_t$ be the set of all s-cells in this dissection; then it is easily seen, by the repeated application of the above lemma, that every s-cycle in V is rationally equivalent to an s-cycle whose support is contained in the bunch $\bigcup_{i=1}^{t} \bar{W}_i$. This means that every s-cycle in V is rationally equivalent to a linear combination of the varieties $\bar{W}_1, \ldots, \bar{W}_t$, considered as prime s-cycles in V, so that these t s-cycles form a rational base for the s-cycles in V. Furthermore, it also follows from the proof of the above lemma that if K is a field of definition for V and the rational dissection such that cycle Z is rational over K, then Z is rationally equivalent over K to a linear combination of the base $\bar{W}_1, \ldots, \bar{W}_t$. In this sense we cay say that the rational base $\bar{W}_1, \ldots, \bar{W}_t$ is defined over any field of definition for the rational dissection of V. Thus we can summarize our results as the following theorem:

THEOREM. *The set of all s-cells in a rational dissection of a variety V constitutes a rational base for the s-cycles in V, which is defined over any field of definition for the rational dissection.*

* This work was partially supported by the Office of Ordnance Research, United States Army.

[1] We shall use in general the terminology of Weil's *Foundations of Algebraic Geometry*. By a "variety" we shall mean an "Abstract Variety" in the sense of Weil, unless the contrary is explicitly stated. All varieties are assumed to be topologized with the Zariski topology, in which the closed subsets are the bunches of subvarieties; this will enable us to use the terminology of topology without any further explanation. The concept of a specialization of a cycle (of arbitrary dimension) in a variety can be defined as in the paper of G. Shimura, *Am. J. Math.*, **77,** 134, 1955.

[2] C. Ehresmann, *Ann. Math.*, **35,** 396, 1934.

[3] W. V. D. Hodge and D. Pedoe, *Methods of Algebraic Geometry*, **2,** 337, 1952.

[4] B. L. van der Waerden, *Math. Ann.*, **115,** 619, 1938; van der Waerden attributes this idea to G. Schaake.

[5] The concept of an equivalence class of cycles under rational equivalence is important in the algebraic theory of vector bundles, as we shall show in a forthcoming paper on this subject.

BINARY CONGRUENCES IN ALGEBRAIC NUMBER FIELDS

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Let P be a prime ideal of norm p^f ($(N(P) = p^f)$ in a finite extension F of the rational field. In this note we shall give formulas for the number of solutions $Q(\rho)$ of the congruence

$$\alpha X^m + \beta Y^m + \rho \equiv 0 \pmod{P^\lambda}, \tag{1}$$

where m and λ are positive integers, $k = (m, p^f - 1) \geqq 2$, $(m, p) = 1$, ρ is an arbitrary integer of F, and α, β are integers of F not divisible by P. In addition, we suppose ρ to be in the form

$$\rho = \theta^t \xi \ (\lambda \geqq t \geqq 0, \ (\xi, P) = 1), \tag{2}$$

where θ is an integer of F of the form $\theta = PC$, $(P, C) = 1$. We observe that t is uniquely determined, while ξ is unique (mod P) if $\lambda > t$.

On the basis of formulas for $Q(\rho)$ contained in Theorems 1, 2, and 3 below, we are led to solvability criteria for congruence (1) stated in Theorems 5, 6, and 8. The last result enables us to show that the congruence

$$\alpha X^m + \beta Y^m + \gamma Z^m + \rho \equiv 0 \pmod{P^\lambda} \tag{3}$$

is solvable for all P of sufficiently large norm, subject to $(\alpha\beta\gamma, P) = 1$ and the restrictions on m stated above. For a precise statement of this result see Theorem 9. In all references to $Q(\rho)$ it will be assumed that the above-mentioned conditions on the coefficients and exponents hold.

The results announced in this note are, for the most part, corollaries of more general theorems proved for congruences (mod P^λ) involving an arbitrary number of mth powers. Proofs of the general results will appear in a more detailed paper to be published elsewhere.

Let us suppose that $\chi(a)$ is a fixed primitive kth-power character (mod P). The generalized Jacobi function (mod P), is defined by

$$\psi(y, z) = \sum_{a+b+1 \equiv 0 \pmod{P}} \chi^y(a)\chi^z(b), \tag{4}$$

the summation ranging over elements a, b of a reduced residue system (mod P) such that $a + b + 1 \equiv 0 \pmod{P}$. For a discussion of generalized Jacobi sums we refer to a paper of Vandiver.[1] A sum involving $\psi(y, z)$, which is important in the subject under discussion, is defined by

$$J(\xi) = \sum_{y, z} \chi^y\left(\frac{\xi}{\alpha}\right)\chi^z\left(\frac{\xi}{\beta}\right)\psi(y, z), \tag{5}$$

where the summation is over y and z of a residue system (mod k), such that $y \not\equiv 0 \pmod{k}$, $z \not\equiv 0 \pmod{k}$, $y + z \not\equiv 0 \pmod{k}$. We place, further,

$$\eta = \eta(\alpha, \beta) = k - 1 \quad \text{or} \ -1, \tag{6}$$

according as $\chi(-\beta/\alpha) = 1$ or $\neq 1$.

The following notation is also useful. For $c \geqq 0$ we place

$$g(c) = \sum_{j=0}^{c-1} p^{j(m-2)} = \frac{p^{fc(m-2)} - 1}{p^{f(m-2)} - 1} \quad \text{or } c, \tag{7}$$

according as m > 2 or $m = 2$. Also, define

$$T(\eta) = (\eta + 1)(p^f - 1). \tag{8}$$

The main results for $Q(\rho)$ are divided into three cases, according as (*a*) $\lambda > t \equiv 0 \pmod m$, (*b*) $\lambda > t \not\equiv 0 \pmod m$, (*c*) $\lambda = t$. While it is possible to write down a single formula for $Q(\rho)$ holding in all cases, such a formula is rather complicated and is not needed in the present discussion. On the basis of the above notation, we state first,

THEOREM 1. *If* $\lambda > t = em$, *then*

$$Q(\rho) = p^{f(\lambda-1)} \{ p^{fe(m-2)} (J(\xi) + p^f - \eta) + g(e)\ T(\eta) \}. \tag{9}$$

COROLLARY 1.1. *If* $\lambda > t = 0$ $((\rho, P) = 1)$, *then*

$$Q(\rho) = p^{f(\lambda-1)}(J(\rho) + p^f - \eta). \tag{10}$$

By means of equations (9) and (10), one may deduce a recursion by which $Q(\rho)$ in Theorem 1 is reduced to the case $e = 0$.

THEOREM 2. *If* ξ *is defined as in congruence* (2), *and if* $\lambda > t = em$, *then*

$$Q(\rho) = p^{fe(m-2)}\, Q(\xi) + p^{f(\lambda-1)}\, g(e)\ T(\eta). \tag{11}$$

Moreover, if one writes $Q(\rho) = Q(\rho, \lambda)$, it follows from equation (10) that $Q(\xi, \lambda) = p^{f(\lambda-1)}\, Q(\xi, 1)$. This reduces the case $t = 0$ to the case of a congruence (mod P). Actually, the result in equation (10) with $\lambda = 1$, namely, $Q(\xi, 1) = J(\xi) + p^f - \eta$, is familiar in the equivalent terminology of equations in a Galois field.[2]

Next we state results for $Q(\rho)$ in the remaining cases (*b*) and (*c*). These results are explicit, in that they are independent of $J(\xi)$.

THEOREM 3. *If* $\lambda > t = em + j$ $(m > j > 0)$, *then*

$$Q(\rho) = p^{f(\lambda-1)}\, g(e + 1)\ T(\eta). \tag{12}$$

THEOREM 4. $(\rho = 0)$. *If* $\lambda = t = em + j$ $(m > j \geqq 0)$, *then*

$$Q(\rho) = \begin{cases} p^{f(\lambda-1)} (p^{f(em-2e+j-1)} + g(e + 1)\ T(\eta)) & \text{if } j > 0, \\ p^{f(\lambda-1)} (p^{f(em-2e+1)} + g(e)\ T(\eta) & \text{if } j = 0. \end{cases} \tag{13}$$

From Theorem 3 we obtain the following solvability criterion for congruence (1) in case (*b*).

THEOREM 5. *If* $\lambda > t \not\equiv 0$ *(mod m), then* $Q(\rho) = 0$ *if and only if* $\chi(-\beta/\alpha) \neq 1$.

The relation in formula (11) leads to the following result.

THEOREM 6. *Suppose* $\lambda > t = em > 0$. *Then, in case* $\chi(-\beta/\alpha) \neq 1$, *it follows that* $Q(\rho) = 0$ *if and only if* $Q(\xi) = 0$; *in the remaining case,* $\chi(-\beta/\alpha) = 1$, *it follows that* $Q(\rho) > 0$ *for every* ρ.

To obtain more precise criteria, in case $\lambda > t \equiv 0 \pmod m$, we require the following estimate for $|J(\xi)|$:

$$|J(\xi)| \leqq 2h\sqrt{p^f}, \qquad 2h = (k - 1)(k - 2). \tag{14}$$

This estimate (which follows from the fact that $|\psi(y, z)| = \sqrt{p^f}$ when y, z, and $y + z$ are all $\not\equiv 0 \pmod{k}$) may be applied to equation 10), to give

THEOREM 7. *If* $t = 0$, *then*

$$|Q(\rho) - p^{f\lambda}| \leqq p^{f(\lambda-1)}(k - 1 + 2h\sqrt{p^f}). \tag{15}$$

COROLLARY 7.1. *If* $t = 0$, *then*

$$|Q(\rho) - p^{f\lambda}| \leqq p^{f(\lambda-1/2)}(k - 1)^2.$$

In case $f = 1$, $\chi(\alpha) = \chi(\beta) = 1$, Corollary 7.1 reduces to a result of I. Chowla[3] for rational congruences (mod p^λ). Since $Q(\xi) > 0$ implies that $Q(\rho) > 0$ when $\lambda > t \equiv 0 \pmod{m}$, Theorem 7 leads to the following solvability criterion.

THEOREM 8. *If* $\lambda > t \equiv 0$ *(mod* m*), then* $Q(\rho) > 0$ *for all prime ideals* P *such that*

$$N(P) = p^f > 2h^2 + k - 1 + 2h\sqrt{h^2 + k - 1} \tag{16}$$

and, in particular, if $p^f > (2h + 1)^2$.

By virtue of Theorem 8, we have

THEOREM 9. *Subject to the restrictions* $(\alpha\beta\gamma, P) = 1$, $(m, p) = 1$, $(m, p^f - 1) = k \geqq 2$, *congruence* (3) *is solvable for every prime ideal* P *whose norm* $N(P) = p^f$ *satisfies relation* (16) *and, in particular, if* $p^f > (2h + 1)^2$.

This result follows from Theorem 8, in case $(\rho, P) = 1$ on placing $Z = 0$ in congruence (3), and in case $(\rho, P) \neq 1$ on placing $Z = 1$. For improved estimates in Theorem 9 it is necessary to consider the case $s = 3$ independently. This will be done in the author's forthcoming paper, referred to above. We remark, finally, that a complete discussion of the special case of quadratic congruences (P odd, $m = 2$ in congruences (1) and (3)) has already appeared in a previous paper of the author.[4]

[1] H. S. Vandiver, these PROCEEDINGS, **36,** 144–151, 1950. Vandiver considers generalized Jacobi sums in an arbitrary number of variables.

[2] *Ibid.* References to other papers on equations in a Galois field are also cited there.

[3] I. Chowla, *Proc. Indian Acad. Sci., A*, **4,** 654–655, 1936.

[4] E. Cohen, *Trans. Am. Math. Soc.*, **75,** 444–470, 1953. The results referred to are contained in Theorems 10, 12, and 13. In Theorem 12 it was shown that congruence (3) of the present paper is always solvable if $m = 2$, P odd, $(\alpha\beta\gamma, P) = 1$. This result is also an immediate corollary of Theorem 9 above, with $m = k = 2$. It should be remarked that $J(\xi) = 0$ in case $m = 2$.

COMBINATORIAL PROBLEMS IN THE THEORY OF GRAPHS. I

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1. *Introduction and Terminology.*—As pointed out by one of us,[1] various problems in statistical mechanics lead in a natural way to combinatorial problems in the theory of linear graphs. A number of these problems have been solved more or less completely by the work of Harary and in the dissertations of Norman and Ford,[2] and a short but connected summary of the results will be given here.

We will follow as much as possible the terminology of König's book.[3] However, a connected graph without an articulation point we will call a "star." A general connected graph is divided by its articulation points into a number of stars.[4] If all stars are just line segments, the connected graphs become the well-known *Cayley trees*. If the stars are polygons, we call the connected graphs *Husimi trees*.[2] If, instead of polygons, more general types of stars are used, we will call the connected graphs *star trees*. A *pure* star tree is one which consists of only one type of star; otherwise, we speak of *mixed* star trees. Clearly, if there are no restrictions on the mixture of types of stars, we have again the general connected graph.

A rooted star tree is a star tree in which one point, the root, is preferred. The *main leaves* of a rooted star tree are the stars which have the root point in common.

The *group* of a graph is the group of the automorphisms of the graph. It can always be considered as a permutation group of the points of the graph. For example, for an n-gon the group is the dihedral group of order $2n$, and for a so-called *complete* graph of n points (i.e., the graph where all pair of points are joined by lines) the group is the symmetric group of degree n. The order of the graph group we will call the *symmetry* number of the graph.

We have considered two types of combinatorial problems. In the first kind of problem, which will be the subject of this note, we ask for the number of different graphs of a given type, in which *all* points are labeled (say, numbered 1, 2, . . .). The second and more difficult kind of problem is to find the corresponding numbers if the points are indistinguishable. We will come back to this in a second note.

2. *The Number of Rooted Star Trees.*—To simplify the writing, we will present the method for *pure* star trees. Let $T(p)$ be the number of rooted pure star trees with p labeled points, in which each star has q points and has the symmetry number s. We introduce the counting series

$$T(x) = \sum_{p=1}^{\infty} \frac{x^p}{p!} T(p).$$

The reason for introducing the $p!$ is that with labeled points we have the following *product property:* The counting series for the number of pairs of graphs selected from two independent graph collections is just the product of the counting series of the separate collections.

Let $T_m(x)$ be the counting series of the rooted star trees with exactly m main leaves. Clearly

$$T(x) = \sum_{m=0}^{\infty} T_m(x), \tag{1}$$

with $T_0(x) \equiv x$. Now we can construct rooted star trees with m mean leaves by selecting m rooted star trees with *one* main leaf and hanging them all on the root. From the product property it follows that

$$T_m(x) = \frac{x}{m!}\left[\frac{T_1(x)}{x}\right]^m \tag{2}$$

since $T_1(x)/x$ counts the rooted star trees of one main leaf in which the root point is erased; the factor x accounts for the root, and the $m!$ is needed because the m root points are in fact identical.

To find $T_1(x)$, we start with the main leaf and hang rooted star trees on the $(q-1)$ points other than the root. Again from the product property,

$$T_1(x) = x \frac{q}{s} T^{q-1}(x). \tag{3}$$

The factor x accounts for the root; the factor q is the number of ways in which we can select the root point, and the factor $1/s$ is needed, since there are s equivalent arrangements of root and rooted star trees on the main leaf because of its symmetry.

Introducing equations (3) and (2) in equation (1), one obtains

$$T(x) = x \exp\left[\frac{q}{s} T^{q-1}(x)\right]. \tag{4}$$

To generalize this result to *mixed* rooted star trees, let $T(p; n_1, n_2, \ldots)$ be the number of rooted star trees having p labeled points and consisting of n_1 stars of type 1, n_2 stars of type 2, etc. For instance, for mixed Husimi trees the different types would be the different polygons. We introduce the counting series

$$T(x; y_1, y_2, \ldots) = \sum_{p=1}^{\infty} \frac{x^p}{p!} \sum_{[n_\sigma]}{}' T(p; n_1, n_2, \ldots) y_1^{n_1} y_2^{n_2}, \ldots, \tag{5}$$

where the round sum goes over all sets $[n_\sigma]$ consistent with the condition (indicated by the prime)

$$\sum_\sigma n_\sigma (q_\sigma - 1) = p - 1, \tag{6}$$

where q_σ is the number of points in a star of type σ. The auxiliary variables $y_1, y_2, \ldots$ are introduced into the counting series (5) to make it possible to distinguish mixed star trees of different composition. It can easily be seen that the introduction of the y_σ does not invalidate the product property. Because of this, the reasoning given above for the pure star trees can be immediately extended to the mixed case. The result corresponding to equation (4) is[5]

$$T(x; y_1, y_2, \ldots) = \exp \sum_\sigma \frac{q_\sigma y_\sigma}{s_\sigma} T^{q_\sigma - 1}(x; y_1, y_2, \ldots), \tag{7}$$

where s_σ is the symmetry number of a star of type σ.

Using Lagrange's theorem,[6] one obtains from this functional equation the final result

$$T(p; n_1, n_2, \ldots) = \frac{p!\, p^{\sum_\sigma n_\sigma - 1}}{\prod_\sigma \left(\frac{s_\sigma}{q_\sigma}\right)^{n_\sigma} n_\sigma!} \tag{8}$$

A special case of this result is that of Husimi,[7] in which the star collection is the collection of complete stars; for the index σ one can use the number of points m in the complete star, and the symmetry number is then $m!$, so that equation (8) becomes[8]

$$\frac{p!\, p^{\sum_2^\infty n_m - 1}}{\prod_2^\infty [(m-1)!]^{n_m} n_m!}, \tag{9}$$

The method used in this section is a generalization of the method used by Polya[9] for the case of rooted Cayley trees, for which equation (8) becomes the well-known result p^{p-1}.

3. *The Number of Free Star Trees.*—For graphs with labeled points the distinction between rooted and free graphs is a trivial one, since each of the labeled points may in turn be preferred as the root. Therefore, if $t(p; n_1, n_2, \ldots)$ is the number of free mixed star trees of p labeled points and of composition $n_1, n_2, \ldots$, then we have

$$T(p; n_1, n_2, \ldots) = p\, t(p; n_1, n_2, \ldots), \tag{10}$$

from which it follows that

$$T(x; y_1, y_2, \ldots) = x \frac{\partial}{\partial x} t(x; y_1, y_2, \ldots), \tag{11}$$

where the counting series $t(x; y_1, y_2, \ldots)$ is defined in terms of the $t(p; n_1, n_2, \ldots)$, analogous to equation (5).

Looking forward to the problem of the number of star trees with indistinguishable points, where the relation between the numbers of rooted and free trees is not so trivial, we will derive another relation between the counting series T and t, which can be generalized to the case of unlabeled points. The relation is obtained by summing equation (6) over all *free* star trees of p points and composition $n_1, n_2, \ldots$; using equation (10) and forming the corresponding counting series, one obtains

$$T(x; y_1, y_2, \ldots) - t(x; y_1, y_2, \ldots) = \sum_\rho (q_\rho - 1) \left[\sum_{p=1}^\infty \frac{x^p}{p!} \frac{S'}{[n_\sigma]}\, n_\rho\, t(p; n_1, n_2, \ldots)\, y_1^{n_1} y_2^{n_2} \ldots \right]. \tag{12}$$

The series in brackets is seen to be the counting series for the number of star trees in which one star of type ρ is given a special designation. This number may also be expressed in terms of the number of rooted trees by starting with the preferred star and counting the number of different ways rooted trees may be hung on the points of the star to form a star tree. Using the product property, this counting series is therefore also equal to

$$\frac{y_\rho}{s_\rho} T^{q_\rho}(x; y_1, y_2, \ldots), \tag{13}$$

where the factor y_ρ accounts for the preferred star of type ρ and the factor $1/s_\rho$ is needed because there are s_ρ equivalent arrangements of the q_ρ rooted star trees on the preferred star.

Inserting expression (13) in equation (12), we arrive at the desired relation

$$t(x; y_1, y_2, \ldots) = T(x, y_1, y_2, \ldots) - \sum_{\rho} \frac{(q_\rho - 1)}{s_\rho} y_\rho T^{q_\rho}(x; y_1, y_2, \ldots). \quad (14)$$

Combining this equation with the relation (11) leads to a differential equation for $T(x)$. The integration of this equation gives a second and independent derivation of the basic functional equation (7) for $T(x; y_1, y_2, \ldots)$. This derivation can be considered as a generalization of the method of Husimi,[3] which in turn was a generalization of the method used by Bol[10] for Cayley trees.

4. *The Numbers of Connected Graphs and of Stars with p Labeled Points and k Lines.*—These numbers, which we will denote by $C(p, k)$ and $S(p, k)$, were first derived by Riddell.[11] His derivation can be simplified considerably by using the product property and equation (7).

Introduce

$$N(p, k) = \binom{\frac{1}{2}p(p-)1}{k}$$

which is the total number of graphs (connected or disconnected) with p labeled points and k lines. Introduce, further, the counting series

$$\begin{aligned} N(x, y) &= \sum_{p=1}^{\infty} \frac{x^p}{p!} \sum_{k=0}^{1/2p(p-1)} N(p, k) y^k, \\ C(x, y) &= \sum_{p=1}^{\infty} \frac{x^p}{p!} \sum_{k=p-1}^{1/2p(p-1)} C(p, k) y^k, \\ S(x, y) &= \sum_{p=1}^{\infty} \frac{x^p}{p!} \sum_{k=p}^{1/2p(p-1)} S(p, k) y^k. \end{aligned} \quad (15)$$

To obtain a relation between $C(x, y)$ and the known $N(x, y)$, write

$$N(x, y) = \sum_{m=1}^{\infty} N_m(x, y), \quad (16)$$

where $N_m(x, y)$ is the counting series for the number of graphs which consist of m disjoint connected parts. From the product property it follows that

$$N_m(x, y) = \frac{1}{m!} C^m(x, y), \quad (17)$$

where the $m!$ is needed because any permutation of the m disjoint parts yields the same graph. From equations (16) and (17) one obtains the first result of Riddell:

$$C(x, y) = \ln [1 + N(x, y)], \quad (18)$$

from which an explicit expression for $C(p, k)$ can be derived.[1]

A relation between $C(x, y)$ and $S(x, y)$ can be obtained from equation (7) by noting that if there are *no* restrictions on the mixture of types of stars, the mixed star trees become the general connected graphs. Therefore, in terms of the counting series $t(x; y_1, y_2, \ldots)$ used in Section 3, one has

$$C(x, y) = t(x; y^{k_1}, y^{k_2}, \ldots), \quad (19)$$

where k_σ is the number of lines in a star of type σ, and σ runs over *all* types of stars. Introducing

$$z(x, y) = x \frac{\partial}{\partial x} C(x, y), \tag{20}$$

it follows from equation (11) and (7) that

$$z(x, y) = x \exp \sum_\sigma \frac{q_\sigma y^{k_\sigma}}{s_\sigma} z^{q_\sigma - 1}(x, y). \tag{21}$$

Now, by arranging the counting series $S(x, y)$ according to the different types of stars σ, it is clear that one may write

$$S(x, y) = \sum_\sigma \left(\frac{q_\sigma!}{s_\sigma}\right) \frac{x^{q_\sigma} y^{k_\sigma}}{q_\sigma!}, \tag{22}$$

since $q_\sigma!/s_\sigma$ is the number of stars with labeled points which are of type σ. The $q_\sigma!$ in the denominator has been put in according to the general prescription for counting series for graphs with labeled points and corresponds to the $p!$ in equations (15). The sum over σ in equation (21) can therefore be written as $(\partial/\partial z)S(z, y)$, and thus one obtains the second relation of Riddell:

$$\frac{\partial}{\partial z} S(z, y) = \ln \frac{z}{x}, \tag{23}$$

where x has to be considered a function of z and y obtained from equation (20). The numbers $C(p, k)$ and $S(p, k)$ may be found for small p by straightforward expansion of equations (18) and (23). Tables of these numbers up to $p = 8$ are given in the dissertation of Riddell,[11] to which we refer also for a discussion of the asymptotic properties of these numbers.[1]

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[1] G. E. Uhlenbeck, *Some Basic Problems of Statistical Mechanics* (Gibbs Lecture, American Mathematical Society, 1950). Compare also R. J. Riddell and G. E. Uhlenbeck, "On the Theory of the Virial Development of the Equation of State of Monoatomic Gases," *J. Chem. Phys.*, **21**, 2056, 1953.

[2] F. Harary and G. E. Uhlenbeck, "On the Number of Husimi Trees. I," these PROCEEDINGS, **39**, 315, 1953; F. Harary, "The Number of Linear Directed, Rooted, and Connected Graphs," *Trans. Am. Math. Soc.*, **78**, 445, 1955; R. Z. Norman, "On the Number of Linear Graphs with Given Blocks" (dissertation, University of Michigan, 1954); G. W. Ford, "On the Theory of Linear Graphs, with Applications to the Theory of Condensation" (dissertation, University of Michigan, 1954).

[3] D. König, *Theorie der endlichen und unendlichen Graphen* (Leipzig, 1936).

[4] König calls these stars the "Glieder" of the connected graph. He has no special term for a graph without articulation points.

[5] The y_σ appears in the sum to account for the main leaf in the rooted mixed star trees with one main leaf of type σ.

[6] E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge, 1927), p. 132.

[7] K. Husimi, "Note on Mayer's Theory of Cluster Integrals," *J. Chem. Phys.*, **18**, 682, 1950. Actually, Husimi obtained the result for free trees, which differs from expression (9) by a factor p (see Sec. 3).

[8] It is not correct to call expression (9) the number of rooted mixed Husimi trees with labeled points, as was done in earlier papers, listed in nn. 1 and 2 above. For Husimi trees equation (8) becomes

$$\frac{p!\, p^{\sum_{3}^{\infty} n_m - 1}}{n_2!\, \prod_{3}^{\infty} 2^{n_m} n_m!}$$

[9] G. Polya, "Kombinatorische Anzahlbestimmungen für Gruppen, Graphen, und chemische Verbindungen," *Acta Math.*, **68**, 145, 1938. See esp. p. 200.

[10] G. Bol, *Abhandl. math. Seminar Hans. Univ.*, **12**, 242, 1938.

[11] R. J. Riddell, "Contributions to the Theory of Condensation" (dissertation, University of Michigan, 1951). See also references listed in n. 2.

NOTE ON PROJECTIVELY EUCLIDEAN HERMITIAN MANIFOLDS

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In a Hermitian manifold M^n of complex dimension n a curve $C: \xi^i = \xi^i(t)$ is called a "geodesic" if the tangent vector $d\xi^i/dt$ at a point remains tangent under a parallel displacement along C. It can be shown that the necessary and sufficient conditions are

$$\frac{d^2\xi^\alpha}{dt^2} + \Gamma^\alpha_{\beta\gamma}\frac{d\xi^\beta}{dt}\frac{d\xi^\gamma}{dt} + 2g_{\gamma\rho^*}g^{\alpha\sigma^*}S_{\tau^*\sigma^*}{}^{\rho^*}\frac{d\xi^{\tau^*}}{dt}\frac{d\xi^\gamma}{dt} = f(t)\frac{d\xi^\alpha}{dt}. \tag{1}$$

We ask whether it is possible to transform a connection Γ^i_{jk} of M^n in such a way that all geodesics remain geodesics. For Kaehler manifolds this was investigated by Bochner,[1] who, moreover, showed that a necessary and sufficient condition for M^n to be projectively Euclidean (flat) is that it have constant holomorphic curvature. In this note, a group of transformations, denoted by G_{SP}, which "preserves" the geodesics is defined, and a generalization of Bochner's result is obtained as follows:[3]

THEOREM. *A manifold is G_{SP}-flat if and only if it has "constant" holomorphic curvature K.*

This result shows that the purely formal definition of such spaces previously given can be motivated in the above manner. Igusa[4] has characterized these manifolds with the Kaehler metric for positive, negative, and zero values of K.

Bochner's result, as well as its generalization, are analogues of a well-known theorem due to Weyl[5] to the effect that a Riemannian manifold is projectively Euclidean if and only if it has constant curvature.

1. The manifold M^n is complex-analytic and is endowed with a Hermitian metric

$$ds^2 = g_{jk}\,dz^j\,dz^k \qquad (j, k = 1, \ldots, n;\ 1^*, \ldots, n^*)$$
$$= 2g_{\alpha\beta^*}\,dz^\alpha\,d\bar{z}^\beta \qquad (\alpha, \beta = 1, \ldots, n), \qquad \bar{z}^\alpha = z^{\alpha^*},$$

where the positive-definite symmetric tensor $g_{jk}(z; \bar{z})$ is self-adjoint and vanishes for indices of opposite parity. The metric connection will be denoted by Γ^i_{jk}, and covariant differentiation with respect to this connection by means of a semicolon,

so that $g_{ij;k} = 0$. We assume that those components of Γ^i_{jk} of opposite parity vanish. The torsion tensor $S_{jk}{}^i = \Gamma^i_{[jk]}$ then has the nonvanishing components $S_{\beta\gamma}{}^\alpha, S_{\beta^*\gamma^*}{}^{\alpha^*}$. The curvature tensor R^i_{jkl} of the metric connection is constructed in the usual manner and has the nonvanishing components $R^\alpha{}_{\beta\gamma\delta^*} = -R^\alpha{}_{\beta\delta^*\gamma}$.

2. Let $\Gamma^i_{jk} \to \Gamma'^i_{jk}$ be a projective transformation with the properties that (*a*) the Γ'^i_{jk} are self-adjoint and vanish for indices of opposite parity and (*b*) the skew-symmetric part of Γ'^i_{jk} coincides with that of Γ^i_{jk}. We call such a mapping a "special projective transformation." If $\Gamma^i_{jk} \to \Gamma^i_{jk} + P^i_{jk}$, the P^i_{jk} are self-adjoint and vanish for indices of opposite parity. Moreover, its skew part vanishes. According to equation (1), we must have

$$P^\alpha_{\beta\gamma} \frac{d\xi^\beta}{dt} \frac{d\xi^\gamma}{dt} = g(t) \frac{d\xi^\alpha}{dt} \tag{2}$$

for every choice of $d\xi^\alpha/dt$ where the function $g(t)$ depends on this choice. However, this is only possible if

$$P^\alpha_{\beta\gamma} = p_\beta \delta^\alpha_\gamma + p_\gamma \delta^\alpha_\beta. \tag{3}$$

It is an easy matter to show that the field p_i is a gradient. The set of all special projective transformations constitutes a group which we denote by G_{SP} and is a subgroup of the group G_P of projective transformations. If the space is Kaehler, $G_{SP} = G_P$. Now

$$\begin{aligned} R'^\alpha{}_{\beta\gamma\delta^*} = \frac{\partial \Gamma'^\alpha_{\beta\gamma}}{\partial \bar{z}^\delta} &= \frac{\partial}{\partial \bar{z}^\delta} (\Gamma^\alpha_{\beta\gamma} + p_\beta \delta^\alpha_\gamma + p_\gamma \delta^\alpha_\beta) \\ &= R^\alpha{}_{\beta\gamma\delta^*} + p_{\beta,\delta^*} \delta^\alpha_\gamma + p_{\gamma,\delta^*} \delta^\alpha_\beta. \end{aligned} \tag{4}$$

A comma is used here to denote partial differentiation.

The tensor

$$P^\alpha{}_{\beta\gamma\delta^*} = R^\alpha{}_{\beta\gamma\delta^*} + \frac{1}{n+1} (R_{\beta\delta^*} \delta^\alpha_\gamma + R_{\gamma\delta^*} \delta^\alpha_\beta) \tag{5}$$

will be called the "projective curvature tensor." It is an easy consequence that it is projectively invariant. Note that it vanishes identically for $n = 1$.

If, under a transformation of the group G_{SP}, the manifold M^n is transformed into a locally Euclidean space, we say that M^n is "G_{SP}-flat." In this case, the tensor $R'^\alpha{}_{\beta\gamma\delta^*}$ vanishes, and

$$R_{\alpha\beta^*\gamma\delta^*} = p_{\alpha,\delta^*} g_{\gamma\beta^*} + p_{\gamma,\delta^*} g_{\alpha\beta^*}.$$

Since $0 = S'_{\beta\gamma}{}^\alpha{}_{,\delta^*} = \Gamma'^\alpha_{[\beta\gamma],\delta^*} = \Gamma^\alpha_{[\beta\gamma],\delta^*} = S_{\beta\gamma}{}^\alpha{}_{,\delta^*}$ and

$$T_{\alpha\beta^*} = (n+1) p_{\alpha,\beta^*}, \qquad T_{\alpha\beta^*} = g^{\gamma\delta^*} R_{\gamma\delta^*\alpha\beta^*},$$

$$S_{\alpha\beta^*} = p_{\alpha,\beta^*} + p^{\gamma^*}{}_{;\,\gamma^*} g_{\alpha\beta^*}, \qquad S_{\alpha\beta^*} = g^{\gamma\delta^*} R_{\alpha\beta^*\gamma\delta^*},$$

it follows that

$$p_{\alpha,\beta^*} = \frac{1}{n} p^{\gamma^*}{}_{;\,\gamma^*} g_{\alpha\beta^*}.$$

Hence

$$R_{\alpha\beta^*\gamma\delta^*} = \frac{K}{2}(g_{\alpha\beta^*}g_{\gamma\delta^*} + g_{\alpha\delta^*}g_{\gamma\beta^*}), \tag{6}$$

where $K = (2/n)p^{\gamma^*}{}_{;\gamma^*}$. Setting $p_{\alpha,\beta^*} = (K/2)g_{\alpha\beta^*}$ (the solution of which is assured) in equation (6), the converse is immediate. In fact,

$$-\frac{\partial^2 \log g}{\partial z^\alpha \partial \bar{z}^\beta} = g^{\gamma\delta^*}R_{\gamma\delta^*\alpha\beta^*} = \frac{(n+1)}{2}Kg_{\alpha\beta^*} = (n+1)p_{\alpha,\beta^*}.$$

COROLLARY 1. *A manifold of "constant" holomorphic curvature can always be mapped projectively (with the geodesics preserved) on a space of the same type.*

COROLLARY 2. *A G_{SP}-flat compact manifold is projectively flat.*

COROLLARY 3. *A G_{SP}-flat manifold is conformal with some Kaehler (Einstein) space.*

Remark: The tensors $G_{\alpha\beta^*\gamma\delta^*}$ and $H_{\alpha\beta^*\gamma\delta^*}$ in Bochner's paper[2] are related: $H_{\alpha\beta^*\gamma\delta^*} = {}^1/_2(G_{\alpha\beta^*\gamma\delta^*} + G_{\gamma\delta^*\alpha\beta^*})$. Hence they need not be treated separately. In fact, $H_{\alpha\beta^*\gamma\delta^*} = 0$ if and only if $G_{\alpha\beta^*\gamma\delta^*} = 0$. Moreover, Theorem 6 in the same paper need only be stated for the tensor G, the statement for the tensor H being redundant.

[1] S. Bochner, "Curvature in Hermitian Metric," *Bull. Am. Math. Soc.*, **53**, 179–195, 1947.

[2] S. Bochner, "Curvature and Betti Numbers. II," *Ann. Math.*, **50** (No. 1), 77–93, 1949.

[3] S. I. Goldberg, "Tensorfields and Curvature in Hermitian Manifolds with Torsion," *Ann. Math.* (to appear).

[4] J. Igusa, "On the Structure of a Certain Class of Kaehler Varieties," *Am. J. Math.*, **76** (No. 3), 669–677, 1954.

[5] H. Weyl, "Zur Infinitesimalgeometrie: Einordnung der projectiven und der konformen Auffassung," *Göttinger Nachr.*, pp. 99–112, 1921.

PREDICATES, TERMS, OPERATIONS, AND EQUALITY IN POLYADIC BOOLEAN ALGEBRAS

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The theory of Boolean algebras is an algebraic counterpart of the logical theory of the propositional calculus; similarly, the theory of polyadic (Boolean) algebras[1] is an algebraic way of studying the first-order functional calculus. The Gödel completeness theorem, for instance, can be formulated in algebraic language as a representation theorem for a large class of simple polyadic algebras, together with the statement that every polyadic algebra is semisimple.[2] The next desideratum is an algebraic study of the celebrated Gödel incompleteness theorem. Before that can be achieved, it is necessary to investigate the algebraic counterparts of some fundamental logical concepts (such as the ones mentioned in the title above). The purpose of this rather technical note is to report (without proofs) the results of such an investigation; the details are being published elsewhere.[3]

1. *Basic Concepts.*—For convenience of reference, this section contains the definitions of the (already known) basic concepts of algebraic logic; all further work is based on these concepts and on their elementary properties.

The following notation will be used for every Boolean algebra $\boldsymbol{A}$: the supremum of two elements p and q of $\boldsymbol{A}$ is $p \vee q$, the infimum of p and q is $p \wedge q$, the complement of p is p', the zero element of $\boldsymbol{A}$ is 0, and the unit element of $\boldsymbol{A}$ is 1. Sometimes it is convenient to write $p \rightarrow q$ instead of $p' \vee q$. The natural order relation is denoted by $\leqq$, so that $p \leqq q$ means that $p \vee q = q$ (or, equivalently, that $p \wedge q = p$ or that $p \rightarrow q = 1$). The simple Boolean algebra $\{0, 1\}$ is denoted by $\boldsymbol{O}$.

A *quantifier* (more precisely, an *existential quantifier*) on a Boolean algebra $\boldsymbol{A}$ is a mapping $\exists$ of $\boldsymbol{A}$ into itself such that (i) $\exists 0 = 0$, (ii) $p \leqq \exists p$, and (iii) $\exists(p \wedge \exists q) = \exists p \wedge \exists q$, for all p and q in $\boldsymbol{A}$. Suppose that $\boldsymbol{A}$ is a Boolean algebra, I is a set, S is a mapping that associates a Boolean endomorphism $\mathsf{S}(\tau)$ of $\boldsymbol{A}$ with every transformation τ from I into I, and $\exists$ is a mapping that associates a quantifier $\exists(J)$ on $\boldsymbol{A}$ with every subset J of I. The quadruple $(\boldsymbol{A}, I, \mathsf{S}, \exists)$ is a *polyadic algebra* if (i) $\exists(\emptyset)$ is the identity mapping on $\boldsymbol{A}$ (here $\emptyset$ is the empty set); (ii) $\exists(J \cup K) = \exists(J)\,\exists(K)$ whenever J and K are subsets of I; (iii) $\mathsf{S}(\delta)$ is the identity mapping on $\boldsymbol{A}$ (here δ is the identity transformation on I); (iv) $\mathsf{S}(\sigma\tau) = \mathsf{S}(\sigma)\,\mathsf{S}(\tau)$ whenever σ and τ are transformations on I; (v) $\mathsf{S}(\sigma)\,\exists(J) = \mathsf{S}(\tau)\,\exists(J)$ whenever $\sigma i = \tau i$ for all i in $I - J$; and (vi) $\exists(J)\,\mathsf{S}(\tau) = \mathsf{S}(\tau)\,\exists(\tau^{-1}J)$ whenever τ is a transformation that never maps two distinct elements of I onto the same element of the set J.

If I is a set, X is a non-empty set, and $\boldsymbol{B}$ is a Boolean algebra, the set of all functions from the Cartesian product X^I into $\boldsymbol{B}$ is a Boolean algebra with respect to the obvious pointwise operations. If τ is a transformation on I, and if x and y are in X^I, write $\tau_* x = y$ whenever $y_i = x_{\tau i}$ for all i in I. (The value of a function x from I into X, i.e., of an element x of X^I, at an element i of I will always be denoted by x_i.) If p is a function from X^I into $\boldsymbol{B}$, write $\mathsf{S}(\tau)p$ for the function defined by $(\mathsf{S}(\tau)p)\,(x) = p(\tau_* x)$. If J is a subset of I and if x and y are in X^I, write $x\ J_*\ y$ whenever $x_i = y_i$ for all i in $I - J$. If p is a function from X^I into $\boldsymbol{B}$, and if the set $\{p(y):\ x\ J_*\ y\}$ has a supremum in $\boldsymbol{B}$ for each x in X^I, write $\exists(J)p$ for the function whose value at x is that supremum. A *functional polyadic algebra* is a Boolean subalgebra $\boldsymbol{A}$ of the algebra of all functions from X^I into $\boldsymbol{B}$, such that $\mathsf{S}(\tau)p \in \boldsymbol{A}$ whenever $p \in \boldsymbol{A}$ and τ is a transformation on I, and such that $\exists(J)p$ exists and belongs to $\boldsymbol{A}$ whenever $p \in \boldsymbol{A}$ and J is a subset of I. The set X is called the *domain* of this functional polyadic algebra.

It is convenient to be slightly elliptical and, instead of saying that $(\boldsymbol{A}, I, \mathsf{S}, \exists)$ is a polyadic algebra, to say that $\boldsymbol{A}$ is a polyadic algebra, or, alternatively, to say that $\boldsymbol{A}$ is an I-algebra. An element of I is called a *variable* of the algebra $\boldsymbol{A}$. The *degree* of $\boldsymbol{A}$ is the cardinal number of the set of its variables. An element p of $\boldsymbol{A}$ is *independent* of a subset J of I if $\exists(J)p = p$; the set J is a *support* of p if p is independent of $I - J$. The algebra $\boldsymbol{A}$ is *locally finite* if each of its elements has a finite support. Some of the results that follow are true for arbitrary polyadic algebras, but most are not. To simplify the statements, it will be assumed throughout the sequel that $(\boldsymbol{A}, I, \mathsf{S}, \exists)$ is a fixed, locally finite polyadic algebra of infinite degree.

Associated with every subset J of I there is a polyadic algebra $(\boldsymbol{A}^-, I^-, \mathsf{S}^-, \exists^-)$ that is said to be obtained from $\boldsymbol{A}$ by *fixing* the variables of J. The set $\boldsymbol{A}^-$ is the

same as the set $\boldsymbol{A}$, and $I^- = I - J$. If τ^- is a transformation on I^-, let τ be its canonical extension to I (i.e., τ is the extension of τ^- to I such that $\tau i = i$ whenever $i \in J$), and write $\mathsf{S}^-(\tau^-)p = \mathsf{S}(\tau)p$ for every p in $\boldsymbol{A}$. If J^- is a subset of I^-, then J^- is also a subset of I; write $\exists^-(J^-)p = \exists(J^-)p$ for every p in $\boldsymbol{A}$.

Suppose that c is a mapping that associates a Boolean endomorphism of $\boldsymbol{A}$ with every subset K of I; denote the value of c at K by $\mathsf{S}(K/c)$. The mapping c is a *constant* of $\boldsymbol{A}$ if (i) $\mathsf{S}(\emptyset/c)$ is the identity mapping on $\boldsymbol{A}$; (ii) $\mathsf{S}(H \cup K/c) = \mathsf{S}(H/c)\,\mathsf{S}(K/c)$, (iii) $\mathsf{S}(H/c)\,\exists(K) = \exists(K)\,\mathsf{S}(H - K/c)$, and (iv) $\exists(H)\,\mathsf{S}(K/c) = \mathsf{S}(K/c)\,\exists(H - K)$, whenever H and K are subsets of I; and (v) $\mathsf{S}(K/c)\,\mathsf{S}(\tau) = \mathsf{S}(\tau)\,\mathsf{S}(\tau^{-1}K/c)$ whenever K is a subset of I and τ is a transformation on I. (The notation here is different from the one used before for constants; the innovation has a beneficial unifying effect.)

2. *Predicates.*—An n-place *predicate* of $\boldsymbol{A}$ ($n = 1, 2, 3, \ldots$) is a function P from I^n into $\boldsymbol{A}$ such that if $(i_1, \ldots, i_n) \in I^n$ and if τ is a transformation on I, then

$$\mathsf{S}(\tau)\,P(i_1, \ldots, i_n) = P(\tau i_1, \ldots, \tau i_n).$$

THEOREM 1. *If P is an n-place predicate of $\boldsymbol{A}$ and if $(i_1, \ldots, i_n) \in I^n$, then $\{i_1, \ldots, i_n\}$ supports $P(i_1, \ldots, i_n)$.*

THEOREM 2. *If $p \in \boldsymbol{A}$ and if $j_1, \ldots, j_n$ are distinct variables such that $\{j_1, \ldots, j_n\}$ supports p, then there exists a unique n-place predicate P of $\boldsymbol{A}$ such that $P(j_1, \ldots, j_n) = p$.*

COROLLARY. *Every element of $\boldsymbol{A}$ is in the range of some predicate of $\boldsymbol{A}$.*

Whereas, by Theorem 1, a value of a predicate depends on no more variables than the visible ones, it may happen that it depends on fewer. If, for example, Q is a 1-place predicate, and if $P(i, j) = Q(i)$ for all i and j, then P is a 2-place predicate such that $\{i\}$ supports $P(i, j)$. It is interesting to ask what happens if the roles of P and Q in this discussion are interchanged. Suppose, in other words, that Q is a 2-place predicate and that j is a particular element of I, and write $P(i) = Q(i, j)$ for all i. The function P is not quite a predicate; it is a $\{j\}$-predicate in the sense of the following definition. If J is a finite subset of I, and if $\boldsymbol{A}^-$ is the algebra obtained from $\boldsymbol{A}$ by fixing the variables of J, then an n-place *J-predicate* of $\boldsymbol{A}$ is an n-place predicate of $\boldsymbol{A}^-$. What was called a predicate before is a $\emptyset$-predicate in the present terminology.

THEOREM 3. *If $j_1, \ldots, j_m$ are distinct elements of I, if $J = \{j_1, \ldots, j_m\}$, and if P is an n-place J-predicate of $\boldsymbol{A}$, then there exists a unique $(n + m)$-place predicate Q of $\boldsymbol{A}$ such that*

$$P(i_1, \ldots, i_n) = Q(i_1, \ldots, i_n, j_1, \ldots, j_m)$$

whenever $(i_1, \ldots, i_n) \in (I - J)^n$.

It is easy to see that the construction described in Theorem 3 always does yield a J-predicate.

3. *Terms.*—Suppose that M and N are finite subsets of I. A transformation σ on I will be said to be of *type* (M, N) if

(i) σ is one-to-one on M,
(ii) $\sigma M \subset I - (M \cup N)$,
and (iii) $\sigma = \delta$ outside M.

If J is a finite subset of I, and if $\boldsymbol{A}^-$ is the algebra obtained from $\boldsymbol{A}$ by fixing the variables of J, then (by definition) a *J-constant* of $\boldsymbol{A}$ is a constant of $\boldsymbol{A}^-$. If c is a J-constant, then $\mathsf{S}(K/c)$ is not defined for every subset K of I; this is the main point in which the concept of a J-term (to be defined next) differs from that of a J-constant. If J is a finite subset of I, a *J-term* is a mapping t that associates with every subset K of I a Boolean endomorphism of $\boldsymbol{A}$, denoted by $\mathsf{S}(K/t)$, so that the restriction of t to the subsets of $I - J$ is a J-constant of $\boldsymbol{A}$, and so that if p is an element of $\boldsymbol{A}$ with finite support L, if K is a finite subset of I, and if σ is a transformation of type $(K, L \cup J)$, then $\mathsf{S}(K/t)p = \mathsf{S}(\sigma K/t)\,\mathsf{S}(\sigma)p$. Every constant is a $\emptyset$-term (and hence a J-term for every finite set J). If $j \in I$ and if t is defined by writing $\mathsf{S}(K/t) = \mathsf{S}(K/j)$ for every subset K of I, then t is a $\{j\}$-term. (The transformation (K/j) is the one that maps every element of K onto j and every element of $I - K$ onto itself.) In this sense, the concept of a term is a simultaneous generalization of the concept of a constant and the concept of a variable.

THEOREM 4. *If c is a J-constant, then there exists a unique J-term t such that $\mathsf{S}(K/t) = \mathsf{S}(K/c)$ whenever $K \subset I - J$.*

4. *Properties of Terms.*—The algebraic behavior of terms is similar to, but necessarily somewhat more complicated than, the algebraic behavior of constants. The chief complication arises in the relation between terms and transformations. To get a concise statement of that relation, it is convenient to introduce a new item of notation. If τ is a transformation on I and if J is a subset of I, then let τ_J be the transformation such that $\tau_J = \tau$ on $\tau^{-1}J$ and $\tau_J = \delta$ outside $\tau^{-1}J$.

THEOREM 5. *If t is a J-term of $\boldsymbol{A}$, if H and K are subsets of I, and if τ is a transformation such that $\tau = \delta$ on J, then*

(T 1) $\mathsf{S}(\emptyset/t)$ *is the identity mapping on* $\boldsymbol{A}$,
(T 2) $\mathsf{S}(H \cup K/t) = \mathsf{S}(H - J/t)\,\mathsf{S}(K - J/t)\,\mathsf{S}((H \cup K) \cap J/t)$,
(T 3) $\mathsf{S}(H/t)\,\mathsf{E}(K) = \mathsf{E}(K - J)\,\mathsf{S}(H - K/t)\,\mathsf{E}(K \cap J)$,
(T 4) $\mathsf{E}(H)\,\mathsf{S}(K/t) = \mathsf{E}(H \cap J)\,\mathsf{S}(K/t)\,\mathsf{E}(H - (J \cap K))$,
(T 5) $\mathsf{S}(K/t)\,\mathsf{S}(\tau) = \mathsf{S}(\tau_{I-J})\,\mathsf{S}((\tau_{I-J})^{-1}K/t)\,\mathsf{S}(\tau_J)$.

THEOREM 6. *If t is a J-term, if K is a subset of I, and if p is an element of $\boldsymbol{A}$ with support L, then $\mathsf{S}(K/t)p = \mathsf{S}(K \cap L/t)p$ and $\mathsf{S}(K/t)p$ has support $J \cup (L - K)$.*

The following result concerning terms is the deepest one; it is used repeatedly in the construction of terms satisfying various conditions.

THEOREM 7. *If J and K are finite subsets of I, if g is a Boolean homomorphism from the range of $\mathsf{E}(K)$ into $\boldsymbol{A}$, and if i is an element of $I - (J \cup K)$ such that* (i) $\mathsf{E}(i)\,g\mathsf{E}(K) = g\mathsf{E}(K)$, (ii) $g\mathsf{E}(K)\,\mathsf{E}(i) = \mathsf{E}(K)\,\mathsf{E}(i)$, (iii) $g\mathsf{E}(K)\,\mathsf{E}(j) = \mathsf{E}(j)\,g\mathsf{E}(K)$ *whenever $j \neq i$ and $j \in I - J$, and* (iv) $g\mathsf{E}(K)\,\mathsf{S}(\tau) = \mathsf{S}(\tau)\,g\mathsf{E}(K)$ *whenever τ is a transformation that maps some finite subset of $I - (\{i\} \cup J \cup K)$ into itself and is equal to δ otherwise, then there exists a J-term t of $\boldsymbol{A}$ such that $\mathsf{S}(i/t)\,\mathsf{E}(K) = g\mathsf{E}(K)$.*

The next result shows that the term t of Theorem 7 is unique.

THEOREM 8. *If J and K are finite subsets of I, if s and t are J-terms, and if i is an element of $I - (J \cup K)$ such that $\mathsf{S}(i/s)p = \mathsf{S}(i/t)p$ whenever p is independent of K, then $s = t$.*

A transformation on I acts in a natural way not only on the variables but also on the terms of $\boldsymbol{A}$. If τ is a transformation and if t is a J-term, let τ_0 be the transformation such that $\tau_0 = \tau$ on J and $\tau_0 = \delta$ outside J, and let i be an element of $I - J$; an

application of Theorems 7 and 8 shows that there exists a unique τJ-term, to be denoted by τt, such that $\mathsf{S}(i/\tau t)p = \mathsf{S}(\tau_0)\ \mathsf{S}(i/t)p$ whenever p is independent of J. This definition of τt is unambiguous (i.e., it does not depend on the particular choice of i).

THEOREM 9. *If t is a J-term, then $\delta t = t$; if σ and τ are transformations, then $(\sigma\tau)t = \sigma(\tau t)$; if $\sigma = \tau$ on J, then $\sigma t = \tau t$.*

If P is an n-place predicate and if $t_1, \ldots, t_n$ are terms, then there is a natural way of defining $P(t_1, \ldots, t_n)$. To do so, suppose that t_1 is a J_1-term, $\ldots$, t_n is a J_n-term, let $i_1, \ldots, i_n$ be distinct elements of $I - (J_1 \cup \ldots \cup J_n)$, and write

$$P(t_1, \ldots, t_n) = \mathsf{S}(i_1/t_1) \ldots \mathsf{S}(i_n/t_n)\ P(i_1, \ldots, i_n).$$

This definition of $P(t_1, \ldots, t_n)$ is unambiguous (i.e., it does not depend on the particular choice of $i_1, \ldots, i_n$).

THEOREM 10. *If P is an n-place predicate, if $t_1, \ldots, t_n$ are terms, and if τ is a transformation, then*

$$\mathsf{S}(\tau)\ P(t_1, \ldots, t_n) = P(\tau t_1, \ldots, \tau t_n).$$

5. *Operations.*—An n-place *operation* ($n = 1, 2, 3, \ldots$) of $\boldsymbol{A}$ is a function T on I^n whose values are terms of $\boldsymbol{A}$, such that if $(i_1, \ldots, i_n) \in I^n$ and if τ is a transformation on I, then

$$\tau T(i_1, \ldots, i_n) = T(\tau i_1, \ldots, \tau i_n).$$

The theory of operations is quite similar to the theory of predicates. All the results of Section 2 have their operation analogues; the following result, for example, is the analogue of Theorem 1.

THEOREM 11. *If T is an n-place operation of $\boldsymbol{A}$ and if $(i_1, \ldots, i_n) \in I^n$, then $T(i_1, \ldots, i_n)$ is an $\{i_1, \ldots, i_n\}$-term.*

The useful part of the theory of operations is based on the fact that there is a natural sense in which the variables of a term can be replaced by terms. (In this rough, descriptive language, the theory of transforms of terms furnishes a natural sense in which the variables of a term can be replaced by other variables.) If s is an H-term, t is a J-term, and K is a subset of I, let i be an element of $I - (H \cup J)$; an application of Theorems 7 and 8 shows that there is a unique $(H \cup (J - K))$-term, to be denoted by $(K/s)t$, such that $\mathsf{S}(i/(K/s)t)p = \mathsf{S}(J \cap K/s)\ \mathsf{S}(i/t)p$ whenever p is independent of J. This definition of $(K/s)t$ is unambiguous.

If T is an n-place operation and if $t_1, \ldots, t_n$ are terms, then there is a natural way of defining $T(t_1, \ldots, t_n)$. To do so, suppose that t_1 is a J_1-term, $\ldots$, t_n is a J_n-term, let $i_1, \ldots, i_n$ be distinct elements of $I - (J_1 \cup \ldots \cup J_n)$, and write

$$T(t_1, \ldots, t_n) = (i_1/t_1) \ldots (i_n/t_n)\ T(i_1, \ldots, i_n).$$

This definition of $T(t_1, \ldots, t_n)$ is unambiguous.

THEOREM 12. *If T is an n-place operation, if $t_1, \ldots, t_n$ are terms, and if τ is a transformation, then*

$$\tau T(t_1, \ldots, t_n) = T(\tau t_1, \ldots, \tau t_n).$$

6. *Equality.*—An *equality* for $\boldsymbol{A}$ is a 2-place predicate E that is *reflexive* (i.e., $E(i, i) = 1$ for all i) and *substitutive* (i.e., $p \wedge E(i, j) = \mathsf{S}(i/j)p \wedge E(i, j)$ for all i and

j and for all p in $\boldsymbol{A}$). (The transformation (i/j) is the one that maps i onto j and everything else onto itself.) An equality E is necessarily *symmetric* (i.e., $E(i, j) = E(j, i)$ for all i and j) and *transitive* (i.e., $E(i, j) \wedge E(j, k) \leqq E(i, k)$ for all i, j, and k). If E is an equality for $\boldsymbol{A}$, then

$$\exists(i)\ (p \wedge E(i, j)) \wedge \exists(i)\ (p' \wedge E(i, j)) = 0$$

whenever $p \in \boldsymbol{A}$ and $i \neq j$, and

$$\exists(i)\ (E(i, j) \wedge E(i, k)) = E(j, k)$$

whenever $i \neq j$ and $i \neq k$. These two equations, together with the other elementary properties of an equality, can be summed up by saying that a polyadic algebra with an equality is a cylindric algebra in the sense of Tarski.

THEOREM 13. *If E is an equality for $\boldsymbol{A}$ and if $\boldsymbol{D}(i, j) = \{p \in \boldsymbol{A}: \mathsf{S}(i/j)p = 1\}$, then the set $\boldsymbol{D}(i, j)$ has an infimum in $\boldsymbol{A}$ for each i and j, and that infimum is equal to $E(i, j)$.*

It is a consequence of Theorem 13 that a polyadic algebra can have at most one equality. The theorem does not say that $\boldsymbol{D}(i, j)$ always has an infimum in $\boldsymbol{A}$ (it does not) or that, if it does, then $\boldsymbol{A}$ must have an equality (it need not). The parenthetical negative assertions are established by the construction of explicit counterexamples.

THEOREM 14. *Every polyadic algebra is a polyadic subalgebra of an algebra with equality.*

It follows from this result that every polyadic algebra can be imbedded into a cylindric algebra. (Here the given algebra may have finite degree, or else it need not even be locally finite; if, however, it is locally finite, then the imbedding algebra can be made to be locally finite also.) The logical counterpart of Theorem 14 asserts the consistency of equality. One way to put it is this: a consistent first-order functional calculus remains consistent when a sign of equality is adjoined to its symbols, and the usual requirements on an equality are adjoined to its axioms.

7. *Algebras with Equality.*—Suppose that I is a set, X is a non-empty set, and $\boldsymbol{B}$ is a Boolean algebra. If i and j are in I, let $E_0(i, j)$ be the function from X^I into $\boldsymbol{B}$ such that $E_0(i, j)\ (x) = 1$ or 0 according as $x_i = x_j$ or $x_i \neq x_j$. The function-valued function E_0 is called the *functional equality* associated with I, X, and $\boldsymbol{B}$. The terminology is justified by the fact that whenever all the values of E_0 (i.e., all the functions $E_0(i, j)$) belong to a $\boldsymbol{B}$-valued functional I-algebra with domain X, then E_0 is an equality for that algebra.

If E is an equality for a $\boldsymbol{B}$-valued functional I-algebra $\boldsymbol{A}$ with domain X, it is not necessarily true that E coincides with the associated functional equality E_0. It is true, however, at least in the important $\boldsymbol{O}$-valued case, that a suitable modification converts E into E_0. The process (an algebraic modification of known logical methods) involves the introduction of an equivalence relation in X induced by E and the reduction of X modulo that relation. The end-product of the process is conveniently described in terms of a new definition: the equality E will be said to be *reduced* if (for all i and j in I and for all x in X^I) $E(i, j)\ (x) = 1$ implies that $x_i = x_j$. The result is that $\boldsymbol{A}$ is isomorphic to a $\boldsymbol{B}$-valued functional I-algebra with a reduced equality. The most useful part of this result is the following special case.

THEOREM 15. *Every* $\mathbf{O}$*-valued functional polyadic algebra with equality is isomorphic to an* $\mathbf{O}$*-valued functional algebra* $\mathbf{A}_0$ *such that the associated functional equality* E_0 *is an equality for* $\mathbf{A}_0$.

These results (together with the known representation theorems for polyadic algebras) imply the algebraic version of the completeness theorem for the first-order functional calculus with equality.

8. *Unique Existence.*—An existential assertion ("there is at least one i such that q") corresponds in algebraic terms to an element of the form $\exists(i)q$. The algebraic correspondent of an assertion of uniqueness ("there is at most one i such that q"), to be denoted by $!(i)q$, is defined (in an algebra with equality E) by

$$!(i)q = \forall(i)\ \forall(j)\ (q \wedge \mathsf{S}(i/j)q \rightarrow E(i, j)).$$

(If J is a subset of I, the operator $\forall(J)$ is defined by $\forall(J)p = (\exists(J)p')'$.) The auxiliary variable j here is to be distinct from i and such that q is independent of j; the definition is easily shown to be independent of the particular choice of j. The algebraic correspondent of an assertion of unique existence ("there is exactly one i such that q"), to be denoted by $\exists!(i)q$, is defined by $\exists!(i)q = \exists(i)q \wedge !(i)q$.

THEOREM 16. *If* J *is a finite set not containing* i, *if* $J \cup \{i\}$ *supports* q, *and if* $\exists!(i)q = 1$, *then there exists a unique* J*-term* t *of* $\mathbf{A}$ *such that* $\mathsf{S}(i/t)q = 1$.

The term t constructed in Theorem 16 ("the i such that q") will be denoted by $\mathsf{I}(i)q$. (The traditional symbol is an inverted iota instead of an inverted T.)

The similarity between the theory of predicates and the theory of terms is clearest in the presence of an equality. There is, in fact, a natural one-to-one correspondence between all operations and some predicates. The relevant predicates are the single-valued ones. An $(n + 1)$-place predicate P is *single-valued* (with respect to its first n arguments) if $\exists!(j)P(i_1, \ldots, i_n, j) = 1$ whenever $(i_1, \ldots, i_n) \in I^n$ and $j \in I - \{i_1, \ldots, i_n\}$.

THEOREM 17. *If* T *is an* n*-place operation and if*

$$P(i_1, \ldots, i_n, j) = E(T(i_1, \ldots, i_n), j)$$

whenever $(i_1, \ldots, i_n, j) \in I^{n+1}$, *then* P *is a single-valued predicate (with respect to its first* n *arguments). If, conversely,* P *is an* $(n + 1)$*-place predicate that is single-valued with respect to its first* n *arguments, then there exists a unique* n*-place operation* T *satisfying the preceding equation; the operation* T *is such that*

$$T(i_1, \ldots, i_n) = \mathsf{I}(j)\ P(i_1, \ldots, i_n, j)$$

whenever $(i_1, \ldots, i_n) \in I^n$ *and* $j \in I - \{i_1, \ldots, i_n\}$.

[1] "Polyadic Boolean Algebras," these PROCEEDINGS, **40**, 296–301, 1954.

[2] "Algebraic Logic. I. Monadic Boolean Algebras," to appear in *Compositio Math.*; "Algebraic Logic. II. Homogeneous Locally Finite Polyadic Boolean Algebras of Infinite Degree," to appear in *Fundamenta Math.*

[3] They will be Parts III and IV of a sequence of papers appearing under the main title "Algebraic Logic."

PROOF OF AN IDENTITY ON JORDAN ALGEBRAS

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1. *Introduction.*—Let $\mathfrak{A}$ be a Jordan algebra over a field $\mathfrak{F}$ of characteristic not 2. Then $\mathfrak{A}$ is a commutative algebra such that

$$x^2(ux) = (x^2u)x \tag{1}$$

for all x, u in $\mathfrak{A}$. As is well known, it follows that

$$[(xy)u]z + [(yz)u]x + [(zx)u]y = (xy)(uz) + (yz)(ux) + (zx)(uy) \tag{2}$$

for all x, y, z, u in $\mathfrak{A}$. Define the *Jordan triple product* $\{a, b, c\}$ for any a, b, c in $\mathfrak{A}$ by

$$\{a, b, c\} = (ab)c + (bc)a - (ca)b, \tag{3}$$

and consider the function

$$\Phi = \{a, b, a\}^2 - \{a, \{b, a^2, b\}, a\}. \tag{4}$$

In a lecture at the University of Chicago last spring N. Jacobson called attention to the fact that the structure of simple Jordan algebras was completely known except for algebras of degree 1. He then showed that this final case could be completed if it were true that the identity $\Phi = 0$ holds for all Jordan algebras. For it follows from his arguments and $\Phi = 0$ that every simple Jordan algebra of degree 1 is one-dimensional. The purpose of this paper is to give a proof of this identity.[1]

2. *Proof of the Identity.*—Let a and b be any two elements of $\mathfrak{A}$. To simplify notation, we will use capitals to denote certain monomials of degree 4 in a and degree 2 in b, as follows:

$$\begin{array}{lll}
A = (a^2b)^2, & F = b[a^2(a^2b)], & L = [a(ab)]^2, \\
B = a^2(a^2b^2), & G = a[(ab)(a^2b)], & M = a\{(ab)[a(ab)]\}, \\
C = a^2(ab)^2, & H = b[(ab)(aa^2)], & N = a^2\{b[a(ab)]\}, \\
D = a^2[b(a^2b)], & J = (ab)[b(aa^2)], & P = (ab)\{a[a(ab)]\}, \\
E = a[b^2(aa^2)], & K = (a^2b)[a(ab)], & Q = a\{a[b(a^2b)]\}.
\end{array} \tag{5}$$

In the following monomials the equalities all follow from equation (1):

$$\begin{array}{llll}
R = b^2(a^2a^2) = b^2[a(aa^2)], & T = b[b(a^2a^2)] = b\{b[a(aa^2)]\}, \\
S = (ab)[a(a^2b)] = (ab)[a^2(ab)], & U = a\{b[a(a^2b)]\} = a\{b[a^2(ab)]\}, \\
V = a[a(a^2b^2)] = a[a^2(ab^2)] = a^2[a(ab^2)], \\
W = b\{a[a(a^2b)]\} = b\{a[a^2(ab)]\} = b\{a^2[a(ab)]\}.
\end{array} \tag{6}$$

We now make substitutions in equation (2), obtaining eleven identities relating the twenty-one monomials (5) and (6) above. The substitution $x = a^2b$, $y = a$, $z = b$, $u = a$ gives

$$W + K + Q = 2S + D. \tag{7}$$

Substitute $x = a$, $y = a$, $z = b$, $u = a$, and multiply both sides of the identity by ab, to get

$$J + 2P = 3S. \tag{8}$$

The substitution $x = a$, $y = a$, $z = b$, $u = a(ab)$ gives

$$W + 2M = N + 2P, \tag{9}$$

and $x = ab$, $y = a$, $z = a$, $u = ab$ gives

$$2M + S = 2L + C. \tag{10}$$

Substituting $x = a$, $y = b$, $z = b$, $u = a$ and multiplying by a^2 give

$$2N + V = 2C + B, \tag{11}$$

and substituting $x = a^2$, $y = a$, $z = b$, $u = a$ and multiplying by b give

$$T + 2W = 2H + F. \tag{12}$$

Substituting $x = a^2$, $y = a^2$, $z = b$, $u = b$ gives

$$T + 2D = R + 2A, \tag{13}$$

Substituting $x = a$, $y = b$, $z = b$, $u = aa^2$ gives

$$2H + E = 2J + R, \tag{14}$$

and substituting $x = a^2b$, $y = a$, $z = a$, $u = b$ gives

$$2U + A = 2S + D. \tag{15}$$

We substitute $x = a$, $y = a$, $z = b$, $u = a^2b$, to obtain

$$F + 2G = D + 2S, \tag{16}$$

and we finally substitute $x = a$, $y = b$, $z = b$, $u = a^2$ and multiply by a, to get

$$2U + V = 2G + E. \tag{17}$$

From equation (3) we see that

$$\Phi = 4L - 4(K + Q) + A + 2V + 2D - B.$$

Thus Φ is a linear combination of seven of the twenty-one formally distinct monomials. We prove that $\Phi = 0$ by a reduction process using the eleven linear equations obtained by substitution, and we are essentially able to complete the proof, since three monomials drop out by themselves.

Our first step is to use equation (7) to write

$$\begin{aligned}\Phi &= 4L - 4(2S + D - W) + A + 2V + 2D - B \\ &= 4L - 8S - 2D + 4W + A + 2V - B.\end{aligned}$$

By equation (8) we replace $6S$ by $2(J + 2P)$ and get

$$\Phi = 4L - 2S - 2(J + 2P) - 2D + 4W + A + 2V - B.$$

Eliminating $2P$ by equation (9) gives

$$\begin{aligned}\Phi &= 4L - 2S - 2J - 2(W + 2M - N) - 2D + 4W + A + 2V - B \\ &= 4L - 2S - 2J - 4M + 2N - 2D + 2W + A + 2V - B.\end{aligned}$$

Eliminating $4M$ by equation (10) yields

$$\begin{aligned}\Phi &= 4L - 2S - 2J - 2(2L + C - S) + 2N - 2D + 2W + A + 2V - B \\ &= -2J - (2C + B) + 2N - 2D + 2W + A + 2V,\end{aligned}$$

where we observe that both L and S have dropped out. Equation (11) allows us to replace $2C + B$ by $2N + V$ and get

$$\begin{aligned}\Phi &= -2J - (2N + V) + 2N - 2D + 2W + A + 2V \\ &= -2J - 2D + 2W + A + V,\end{aligned}$$

and N has now dropped out. By equation (12) we have

$$\begin{aligned}\Phi &= -2J - 2D + (2H + F - T) + A + V \\ &= -2J + 2H + F - (T + 2D) + A + V,\end{aligned}$$

and by equation (13),

$$\begin{aligned}\Phi &= -2J + 2H + F - (R + 2A) + A + V \\ &= -(2J + R) + 2H + F - A + V.\end{aligned}$$

By equation (14),

$$\begin{aligned}\Phi &= -(2H + E) + 2H + F - A + V \\ &= -E + F - A + V,\end{aligned}$$

and H has been eliminated. By equation (15),

$$\Phi = -E + F - (2S + D - 2U) + V,$$

and S has been reinserted. However, by equation (16), $2S + D = F + 2G$, from which

$$\begin{aligned}\Phi &= -E + F - (F + 2G) + 2U + V \\ &= 2U + V - (2G + E).\end{aligned}$$

It follows at once from equation (17) that $\Phi = 0$, and the proof is complete.

[1] Marshall Hall, Jr., also has a proof of the identity, which will be published in another journal. He obtains the result by determining a basis for the polynomials of low degree in the free Jordan ring with two generators.

*A THEOREM ON THE STRUCTURE OF JORDAN ALGEBRAS**

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The purpose of this paper is to fill a gap in Albert's structure theory[1] of abstract Jordan algebras of any characteristic $\neq 2$, by proving the following:

Theorem A. *Let $\mathfrak{A}$ be a finite-dimensional Jordan algebra over a field Φ. Assume (1) that $\mathfrak{A}$ has an identity element u and (2) that every element of $\mathfrak{A}$ has the form $\alpha u + z$, where $\alpha \in \Phi$ and z is nilpotent. Then $\mathfrak{A} = \Phi u + \mathfrak{B}$, where $\mathfrak{B}$ is a nil subalgebra of $\mathfrak{A}$.*

This result was proved by Albert for special Jordan algebras and was used by him as a key result in the structure theory of semisimple commutative power-associative algebras. A question which was left open in Albert's work was that of the structure of simple Jordan algebras over an algebraically closed field having only one nonzero idempotent, the identity u.[2] Theorem A, as stated, implies easily that such an algebra is necessarily Φu. This result completes the classification of simple Jordan algebras over an arbitrary field. It also permits us to fill several gaps in the representation theory of Jordan algebras which has been developed by the present author.[3]

Our proof will be based on two new concepts—inverses and ternary composition—which appear to be of some intrinsic interest. Our results on these enable us to adapt Albert's proof of the theorem for special Jordan algebras to the abstract case. An essential step in the proof is a certain ternary identity which we formulated and which has been proved independently by M. Hall and by L. R. Harper, Jr. (their proofs will appear in a forthcoming issue of the *Proc. Am. Math. Soc.*).

1. INVERSES

A (nonassociative) algebra $\mathfrak{A}$ is called a *Jordan algebra* if its multiplication satisfies the identities

$$ab = ba, \qquad (a^2b)a = a^2(ba). \tag{1}$$

It is always assumed that the characteristic of the base field is $\neq 2$. If $\mathfrak{A}$ is a subspace of an associative algebra $\mathfrak{E}$, and $\mathfrak{A}$ is closed relative to the composition $\{ab\} \equiv ab + ba$, then $\mathfrak{A}$ with this composition as multiplication is a Jordan algebra. Jordan algebras of this type are called *special*.

Identities (1) have a number of well-known consequences,[4] which we list as follows:

$$((ab)d)c + ((bc)d)a + ((ca)d)b = (ab)(dc) + (bc)(da) + (ca)(db). \tag{2}$$

We write $A(x, y, z)$ for the *associator* $(xy)z - x(yz)$. Then equation (2) can be rewritten as

$$A(ab, d, c) + A(bc, d, a) + A(ca, d, b) = 0. \tag{2'}$$

We write R_a for the linear mapping $x \to xa$ in $\mathfrak{A}$. Then

$$\begin{aligned} R_aR_bR_c + R_cR_bR_a + R_{(ac)b} &= R_aR_{bc} + R_bR_{ac} + R_cR_{ab} \\ &= R_{bc}R_a + R_{ac}R_b + R_{ab}R_c, \end{aligned} \tag{3}$$

$$[[R_c R_b]R_a] = R_{A(a,b,c)}, \qquad ([XY] \equiv XY - YX). \tag{4}$$

We define $a^1 = a$, $a^k = a^{k-1}a$. Then $\mathfrak{A}$ is *power associative* in the sense that $a^k a^l = a^{k+l}$. Also, we have

$$R_{a^n} = 2R_a R_{a^{n-1}} - (2R_{a^2} - R_{a^2})R_{a^{n-2}}, \tag{5}$$

$$R_{a^n} = \frac{1}{2}\left\{(R_a + (R_{a^2} - R_{a^2})^{1/2})^n + (R_a - (R_{a^2} - R_{a^2})^{1/2})^n\right\}. \tag{6}$$

Let $\mathfrak{A}$ be a special Jordan subalgebra of an associative algebra $\mathfrak{E}$ with an identity element 1. If $1 \in \mathfrak{A}$, then $u = {}^1/_2\, 1$ acts as the identity for $\mathfrak{A}$ relative to the composition $\{ab\} = ab + ba$. If a and $b \in \mathfrak{A}$ and $ab = 1 = ba$, then

$$\{ab\} = 4u, \qquad \{\{aa\}b\} = 4a, \qquad \{\{aa\}\{bb\}\} = 16u. \tag{7}$$

Conversely, it is easy to see that these Jordan relations imply that $ab = 1 = ba$. This leads to the following definition.

Definition 1: Let $\mathfrak{A}$ be a Jordan algebra with an identity u. Then an element $a \in \mathfrak{A}$ is called *regular* in $\mathfrak{A}$ if there exists a $b \in \mathfrak{A}$ such that

$$ab = 4u, \qquad a^2b = 4a, \qquad a^2b^2 = 16u. \tag{8}$$

The element b is called an *inverse* of a.

If a is regular with b as an inverse, then $ba = 4u$ and $b^2a = \frac{1}{4}(a^2b)b^2 = \frac{1}{4}(a^2b^2)b = 4b$. Hence b is regular with a as an inverse. If we set $c = \frac{1}{2}a$, $d = \frac{1}{2}b$, then the defining relations give $cd = u$, $c^2d = c$, $c^2d^2 = u$. Using these and equation (5), we can establish

$$c^l d^k = \begin{cases} c^{l-k} & \text{if } l > k, \\ u & \text{if } l = k, \\ d^{k-l} & \text{if } l < k. \end{cases} \tag{9}$$

These show that the subalgebra $\Phi[u, a, b]$ generated by u, a, b is a homomorphic image of the group algebra of an infinite cyclic group. In particular, $\Phi[u, a, b]$ is an associative subalgebra of $\mathfrak{A}$.

We take up next the questions of uniqueness of the inverse. For this purpose we introduce

Definition 2: If $\mathfrak{A}$ is a Jordan algebra, then we define the *Jordan triple product* $\{abc\} \equiv \frac{1}{2}(ab)c + \frac{1}{2}(bc)a - \frac{1}{2}(ac)b$, so that $\{aba\} = (ab)a - \frac{1}{2}a^2b = bU_a$, where $U_a \equiv R_a{}^2 - \frac{1}{2}R_{a^2}$.

If $\mathfrak{A}$ is a special Jordan algebra, then $\{abc\} = abc + cba$ in terms of the associative multiplication. Hence $\{aba\} = 2aba$. A fundamental property of regular elements, which we proceed to establish, is that if a is regular, then the linear operator U_a is regular (has an inverse).

LEMMA 1. *Let $\mathfrak{B}$ be an associative Jordan subalgebra of a Jordan algebra $\mathfrak{A}$, and let $R(\mathfrak{B})^*$ be the associative algebra of linear transformations generated by the R_b (acting in $\mathfrak{A}$), with $b \in \mathfrak{B}$. Then $R(\mathfrak{B})^*$ contains a nil ideal $\mathfrak{N}$ such that $R(\mathfrak{B})^*/\mathfrak{N}$ is commutative.*

Proof: We are assuming that if $a, b, c \in \mathfrak{B}$, then $A(a, b, c) = 0$. By formula (4), this implies that $[[R_c R_a]R_b] = 0$ holds in $R(\mathfrak{B})^*$. Now consider the relation $2R_aR_bR_a = -R_{a^2b} + R_{a^2}R_b + 2R_{ab}R_a$. Take the commutators with R_b. Since this process is a derivation, we obtain

$$2[R_aR_b]R_bR_a + 2R_a[R_aR_b] = -[R_{a^2b}R_b] + [R_{a^2}R_b]R_b + 2[R_{ab}R_b]R_a + 2R_{ab}[R_aR_b].$$

Take commutators with R_b again, and use the fact that commutators are in the center. This gives

$$4[R_aR_b]^2R_b = 4[R_{ab}R_b][R_aR_b].$$

If we take the commutators with R_a, this gives $4[R_aR_b]^3 = 0$ and $[R_aR_b]^3 = 0$. Hence the commutators $[R_aR_b]$ are nilpotent. Since they are in the center, they generate a nil ideal $\mathfrak{N}$ in $R(\mathfrak{B})^*$. Since $R_aR_b \equiv R_bR_a \pmod{\mathfrak{N}}$ and the R_a generate $R(\mathfrak{B})^*$, $R(\mathfrak{B})^*/\mathfrak{N}$ is commutative.

THEOREM B. *Let $\mathfrak{A}$ be a Jordan algebra with an identity u, and let a be a regular element of $\mathfrak{A}$ with b as an inverse. Then $U_a = R_a^2 - \frac{1}{2}R_{a^2}$ has an inverse in the algebra of linear transformations of $\mathfrak{A}$. Moreover, if the characteristic is $\neq 3$, then $U_a^{-1} = \frac{1}{4}U_b$.*

Proof: Let $\mathfrak{B}$ be the subalgebra generated by u, a, and b, and let $R(\mathfrak{B})^*$ be the associative algebra generated by the linear transformations $R_u = 1$, R_x for $x \in \mathfrak{B}$. Then $R(\mathfrak{B})^*$ contains a nil ideal $\mathfrak{N}$ such that $R(\mathfrak{B})^*/\mathfrak{N}$ is commutative. If we use relations (8) in $R_{a^2b} + 2R_aR_bR_a = R_bR_{a^2} + 2R_aR_{ab} = R_{a^2}R_b + 2R_{ab}R_a$, we obtain $4R_a + 2R_aR_bR_a = R_bR_{a^2} + 8R_a = R_{a^2}R_b + 8R_a$. Hence

$$[R_{a^2}R_b] = 0, \qquad 4R_a = 2R_aR_bR_a - R_{a^2}R_b. \tag{10}$$

By symmetry, $[R_{b^2}R_a] = 0$. Next, use relations (8) in $R_{a^2b^2} + 2R_aR_{b^2}R_a = R_{b^2}R_{a^2} + 2R_aR_{ab^2}$, to obtain

$$16 = R_{b^2}R_{a^2} + 8R_aR_b - 2R_aR_{b^2}R_a. \tag{11}$$

If we multiply the second part of equations (10) by $2R_b$ on the right-hand side and substitute in equation (11), we obtain

$$16 = R_{b^2}R_{a^2} + 4(R_aR_b)^2 - 2R_{a^2}R_b^2 - 2R_aR_{b^2}R_a. \tag{12}$$

If we use the fact that $R(\mathfrak{B})^*/\mathfrak{N}$ is commutative, equation (12) implies

$$16 \equiv (R_{a^2} - 2R_a^2)(R_{b^2} - 2R_b^2) \pmod{\mathfrak{N}}. \tag{13}$$

This shows that $R_{a^2} - 2R_a^2$ has an inverse mod $\mathfrak{N}$, and, since $\mathfrak{N}$ is a nil ideal, it implies that $R_{a^2} - 2R_a^2$, and hence U_a has an inverse in $R(\mathfrak{B})^*$. We now use the relation $[R_{xy}R_z] + [R_{yz}R_x] + [R_{zx}R_y] = 0$ (cf. Jacobson[3]), with $x = a^2$, $y = a$, $z = b$, and relations (8), to obtain $[R_{a^2}R_b] = 0$. Since $R_{a^3} = 3R_aR_{a^2} - 2R_a^3$ and $[R_{a^2}R_b] = 0$, this gives $3R_{a^2}[R_aR_b] - 6R_a^2[R_aR_b] = 0$. If the characteristic is $\neq 3$, then $U_a[R_aR_b] = 0$, and, since U_a^{-1} exists, $[R_aR_b] = 0$. Since a^2 is regular with $\frac{1}{4}b^2$ as an inverse, a similar argument shows that $[R_{a^2}R_{b^2}] = 0$. Hence the four elements R_a, R_b, R_{a^2}, R_{b^2} commute. By equation (6), these and 1 generate $R(\mathfrak{B})^*$.

Hence $R(\mathfrak{B})^*$ is commutative, so that we may take $\mathfrak{N} = 0$. Then relation (13) gives $16 = (R_{a^2} - 2R_a^2)(R_b^2 - 2R_b^2)$, so that $U_a(\frac{1}{4}U_b) = 1$.

We shall say that a is a *zero divisor* if there exists a $b \neq 0$ such that $bU_a = 0$. Theorem B shows that if a is regular, then a is not a zero divisor. Now let a be regular, and let b and b' be inverses. Then $bU_a = 2a = b'U_a$, so that $(b - b')U_a = 0$. Hence $b = b'$. This proves the uniqueness of the inverse.

If $\mathfrak{A}$ is an associative Jordan algebra (commutative associative algebra) with an identity, then a is regular (a zero divisor) in $\mathfrak{A}$ if and only if a is regular (a zero divisor) in the usual associative sense. Now suppose that $\mathfrak{A}$ is arbitrary and that a is an algebraic element of $\mathfrak{A}$. Then it is well known that a is either regular or a zero divisor in the associative algebra $\Phi[u, a]$ generated by a and u. Hence a is either regular or a zero divisor (in the Jordan sense) in $\mathfrak{A}$. In particular, suppose that $\mathfrak{A} = \Phi u + \mathfrak{B}$, where $\mathfrak{B}$ is a nil subalgebra. Then every $a = \alpha u + b$, $b \in \mathfrak{B}$, is algebraic, and a has an inverse or is nilpotent in the associative algebra $\Phi[u, a]$, according to whether $\alpha \neq 0$ or $\alpha = 0$. In other words, the elements of $\mathfrak{A}$ are either regular or nilpotent, and $\mathfrak{B}$ is just the set of nonregular elements of $\mathfrak{A}$.

2. JORDAN TRIPLE PRODUCT IDENTITIES

In this section we list some useful properties of the Jordan triple product $\{abc\} = \frac{1}{2}(ab)c + \frac{1}{2}(bc)a - \frac{1}{2}(ac)b$. Evidently, $\{abc\}$ is trilinear, that is, is linear in any one of the arguments if the other two are fixed. Also, we have

$$\{abc\} = \{cba\}, \tag{14}$$

$$\{abc\} + \{bac\} = (ab)c. \tag{15}$$

The last has the consequence

$$\{bac\} - \{acb\} = A(a, b, c). \tag{16}$$

We now list

$$\{aa^ka\} = \frac{1}{2}a^{k+2}, \qquad k \geq 1, \tag{17}$$

$$\{a\{a^kba^k\}a\} = \frac{1}{2}\{a^{k+1}ba^{k+1}\}, \qquad k \geq 1, \tag{18}$$

$$\{aba\}b = a\{bab\}, \tag{19}$$

$$\{(ca)ba\} - \frac{1}{2}\{a(bc)a\} = \frac{1}{2}\{aba\}c, \tag{20}$$

$$(((ab)a)b) = \frac{1}{4}a^2b^2 + \frac{1}{2}\{ba^2b\} + \{aba\}b, \tag{21}$$

$$\frac{1}{2}\{ab^2a\} + \frac{1}{2}\{ba^2b\} + a\{bab\} = (ab)^2. \tag{22}$$

Proofs: Equation (17): $\{aa^ka\} = \frac{1}{2}a^{k+2} + \frac{1}{2}a^{k+2} - \frac{1}{2}a^{k+2} = \frac{1}{2}a^{k+2}$. Equation

(18): Using equation (6), we obtain

$$(R_a{}^k)^2 = \frac{1}{4}\{R_a + (R_{a^2} - R_{a^2})\} + {}^{2k}\frac{1}{4}\{R_a - (R_{a^2} - R_{a^2})\}^{2k} + \frac{1}{2}(R_{a^2} - (R_{a^2} - R_{a^2}))^k$$

$$= \frac{1}{2}R_{a^{2k}} + 2^{k-1}U_a{}^k.$$

Since $\{a^kba^k\} = bU_a{}^k$, this gives $\{a^kba^k\} = 2^{k-1}bU_a{}^k$. Hence $\{a\{a^kba^k\}a\} = 2^{k-1}bU_a{}^{k+1} = \frac{1}{2}2^kbU_a{}^{k+1} = \frac{1}{2}\{a^{k+1}ba^{k+1}\}$. Equation (19): Subtraction of $2A(ab, b, a) + A(a^2, b, b) = 0$ from $2A(ab, a, b) + A(b^2, a, a) = 0$ gives $2((ab)a)b - 2(ab)^2 + (ab^2)a - a^2b^2 - 2((ab)b)a + 2(ab)^2 - (a^2b)b + a^2b^2 = 2((ab)a)b - (ba^2)b - 2((ab)b)a + (ab^2)a = 0$. Hence $\{aba\}b = \{bab\}a$. Equation (20): $2\{(ca)ba\} - \{a(bc)a\} - \{aba\}c = ((ca)b)a + (ba)(ca) - ((ca)a)b - ((bc)a)a + \frac{1}{2}(bc)a^2 - ((ba)a)c + \frac{1}{2}(ba^2)c = A(b, ca, a) - A(ba, a, c) - A(bc, a, a) + \frac{1}{2}A(a^2, b, c) = A(b, ca, a) + A(ca, a, b) - A(ca, b, a)$, since $\frac{1}{2}A(a^2, b, c) = -A(ca, b, a)$ and $-A(ba, a, c) - A(bc, a, a) = A(ca, a, b)$ by relation (2′). Now $A(b, ca, a) + A(ca, a, b) - A(ca, b, a) = A(b, ca, a) + A(ca, a, b) + A(a, b, ca) = 0$. Hence $2\{\{ca\}ba\} = \{a(bc)a\} + \{aba\}c$. Equation (21): $2\{aba\}b + \{ba^2b\} + \frac{1}{2}a^2b^2 = 2((ab)a)b - (ba^2)b + (a^2b)b - \frac{1}{2}a^2b^2 + \frac{1}{2}a^2b^2 = 2((ab)a)b$. Equation (22): $\{ab^2a\} + \{ba^2b\} + 2a\{bab\} - (ab)^2 = (b^2a)a - \frac{1}{2}b^2a^2 + (a^2b)b - \frac{1}{2}a^2b^2 + 2((ab)b)a - (ab^2)a - 2(ab)^2 = (a^2b)b - a^2b^2 + 2((ab)b)a - 2(ab)^2 = A(a^2, b, b) + 2A(ab, b, a) = 0$.

The following identity will be proved in forthcoming papers by M. Hall and L. R. Harper, Jr.:

$$\{aba\}^2 = \{a\{ba^2b\}a\}. \tag{23}$$

If we linearize relation (23) with respect to b, we obtain $\{aba\}\{aca\} = \{a\{ba^2c\}a\}$, which shows that the image space $\mathfrak{A}U_a$ of $\mathfrak{A}$ is a subalgebra of $\mathfrak{A}$.

Though we shall not require these, we list two more identities which can easily be established for special Jordan algebras:

$$\{\{aba\}b\{abc\}\} = \{\{\{aba\}bc\}ba\}, \tag{24}$$

$$\{\{aba\}c\{aba\}\} = \{a\{b\{aca\}b\}a\}. \tag{25}$$

We conjecture that every free Jordan algebra is special. This would mean that every Jordan algebra is a homomorphic image of a special one and that every identity valid for all special Jordan algebras is valid for all Jordan algebras. However, at the present time we do not know whether relations (24) and (25) hold for all Jordan algebras.

3. PROOF OF THEOREM A

LEMMA 2. *Let $\mathfrak{A}$ be a Jordan algebra with an identity u, and let z be a nilpotent element of $\mathfrak{A}$. Then $\{zaz\}$ is not regular for any $a \in \mathfrak{A}$.*

Proof: We may assume that $z^{r-1} \neq 0$, $z^r = 0$ $r \geq 2$. Consider $\{\{zaz\}z^{r-1}\{zaz\}\} = (z^{r-1}\{zaz\})\{zaz\} - z^{r-1}\{zaz\}^2$. We have $z^{r-1}\{zaz\} = a(R_z{}^2 - \frac{1}{2}R_{z^2})R_{z^{r-1}} = a\left(-\frac{1}{2}R_{z^{r+1}} + R_zR_{z^r}\right) = 0$. Similarly, since, by relation (23), $\{zaz\}^2$ has the form $\{zbz\}$, $\{zaz\}^2z^{r-1} = 0$. Hence $\{\{zaz\}z^{r-1}\{zaz\}\} = 0$. Since $z^{r-1} \neq 0$, this implies that U_w has no inverse for $w = \{zaz\}$. Hence, by Theorem B, $\{zaz\}$ is not regular.

Suppose, now, that $\mathfrak{A}$ is a Jordan algebra satisfying the hypotheses of Theorem A. Let $\mathfrak{C}$ be a subalgebra of the form $\Phi u + \mathfrak{B}$, where $\mathfrak{B}$ is a nil subalgebra. Suppose that $\mathfrak{C} \neq \mathfrak{A}$. It is known that the associative algebra generated by the elements R_b, $b \in \mathfrak{B}$, is nilpotent.[5] This implies that there exists an element $x \notin \mathfrak{C}$ such that $xb \in \mathfrak{C}$ for every $b \in \mathfrak{B}$. By subtracting a suitable multiple of u from x, we may suppose that x is nilpotent.

LEMMA 3. *$\{bxb\}$, xb^2, x^2b^2, $\{bx^2b\}$, $\{xb^2x\} \in \mathfrak{B}$, and $\{bxb\}x$, $b'\{xbx\} \in \mathfrak{C}$, for every b, $b' \in \mathfrak{B}$.*

Proof: $\{bxb\} = (xb)b - \frac{1}{2}xb^2 \in \mathfrak{C}$. By Lemma 2, $\{bxb\}$ is not regular. Hence $\{bxb\} \in \mathfrak{B}$. Since $xb \in \mathfrak{C}$, and $\mathfrak{B}$ is an ideal in $\mathfrak{C}$, $(xb)b \in \mathfrak{B}$. Hence $xb^2 \in \mathfrak{B}$. Set $xb = \beta u + b'$, b' in $\mathfrak{B}$. Then, by relation (22), $(xb)^2 = \frac{1}{2}\{xb^2x\} + \frac{1}{2}\{bx^2b\} + x\{bxb\} \in \mathfrak{C}$. Since $x\{bxb\} \in \mathfrak{C}$, this gives

$$\{xb^2x\} + \{bx^2b\} \in \mathfrak{C}. \tag{26}$$

Also, $x(b^2x) \in \mathfrak{C}$ and $x(b^2x) = \{xb^2x\} + \frac{1}{2}x^2b^2$. Hence

$$\{xb^2x\} + \frac{1}{2}x^2b^2 \in \mathfrak{C}. \tag{27}$$

Next, note that $(xb)x = \beta x + b'x = \beta x + c$, c in $\mathfrak{C}$. Also, $x^2b = -2\{xbx\} + 2(bx)x$, so that $(x^2b)b = -2\{xbx\}b + 2((bx)x)b = -2x\{bxb\} + 2(\beta x + c)b = -2x\{bxb\} + 2\beta^2u + b''$, b'' in $\mathfrak{B}$. Hence

$$(x^2b)b = -2x\{bxb\} + 2\beta^2u + b'', \qquad b'' \in \mathfrak{B}. \tag{28}$$

Since $(x^2b)b = \{bx^2b\} + \frac{1}{2}x^2b^2$, this shows that

$$\{bx^2b\} + \frac{1}{2}x^2b^2 \in \mathfrak{C}. \tag{29}$$

Relations (26), (27), and (29) imply that $\{xb^2x\}$, $\{bx^2b\}$, $x^2b^2 \in \mathfrak{C}$. Since x and b are nilpotent, $\{xb^2x\}$ and $\{bx^2b\}$ are not regular. Hence they are in $\mathfrak{B}$. By equation (28),

$$x\{bxb\} = -\frac{1}{2}(x^2b)b + \beta^2u + \frac{1}{2}b'', \tag{30}$$

and, by equation (22),

$$\begin{aligned} x\{bxb\} &= (xb)^2 - \frac{1}{2}\{xb^2x\} - \frac{1}{2}\{bx^2b\} \\ &= \beta^2u + 2\beta b' + (b')^2 - \frac{1}{2}\{xb^2x\} - \frac{1}{2}\{bx^2b\}. \end{aligned} \tag{31}$$

Comparison of equations (30) and (31) gives

$$(x^2b)b = \{xb^2x\} + \{bx^2b\} + b''', \qquad b''' \in \mathfrak{B}.$$

This implies that $(x^2b)b \in \mathfrak{B}$. Then $x^2b^2 = 2(x^2b)b - 2\{bx^2b\} \in \mathfrak{B}$. Since $\{bxb\} \in \mathfrak{B}$, $\{bxb\}x \in \mathfrak{C}$. By equation (20),

$$\frac{1}{2}b'\{xbx\} = \{(b'x)bx\} - \frac{1}{2}\{x(b'b)x\}. \tag{32}$$

We have $b'x = \gamma u + \bar{b}$, $\bar{b}$ in $\mathfrak{B}$. Hence

$$\{(b'x)bx\} = \gamma\{ubx\} + \{\bar{b}bx\} = \frac{1}{2}\gamma bx + \{\bar{b}bx\}. \tag{33}$$

Also, by equation (15), $\{\bar{b}bx\} = \bar{b}(bx) - \{\bar{b}xb\}$ and $\bar{b}(bx) \in \mathfrak{B}$. Since $\{bxb\} \in \mathfrak{B}$, linearization shows that $\{\bar{b}xb\} = \frac{1}{2}\{\bar{b}xb\} + \frac{1}{2}\{bx\bar{b}\} \in \mathfrak{B}$. Hence $\{\bar{b}bx\} \in \mathfrak{B}$, and equation (33) shows that $\{(b'x)bx\} \in \mathfrak{C}$. Linearization shows also that $\{x(b'b)x\} \in \mathfrak{B}$. Hence, by equation (32), $b'\{xbx\} \in \mathfrak{C}$.

LEMMA 4. *If $\mathfrak{A} \neq \mathfrak{C}$, then we can find an $x \notin \mathfrak{C}$ such that x is nilpotent and xb and $x^2b \in \mathfrak{B}$ for every $b \in \mathfrak{B}$.*

Proof: Choose x as in Lemma 3. Suppose that $\{xbx\} \in \mathfrak{C}$ for every $b \in \mathfrak{B}$. Since x is nilpotent, it will follow that $\{xbx\} \in \mathfrak{B}$ for every b. Then $(xb)^2 = \frac{1}{2}\{xb^2x\} + \frac{1}{2}\{bx^2b\} + \{xbx\}b \in \mathfrak{B}$. It follows that $b' = xb \in \mathfrak{B}$ for every b in $\mathfrak{B}$. Hence $(xb)x = xb' \in \mathfrak{B}$, and $bx^2 = 2(xb)x - 2\{xbx\} \in \mathfrak{B}$. Thus x satisfies our conditions if $\{xbx\} \in \mathfrak{C}$ for all b. Suppose, next, that there is a b in $\mathfrak{B}$ such that $\{xbx\} \notin \mathfrak{C}$. Since x is nilpotent, $\{xbx\}$ is not regular, and since $\{xbx\} = \delta u + z$, z nilpotent, $\delta = 0$ and $\{xbx\}$ is nilpotent. Also, by Lemma 3, $\{xbx\}b' \in \mathfrak{C}$ for every b'. Hence we may replace x by $\{xbx\}$ in our considerations. Since $\{xbx\} \notin \mathfrak{C}$ and $\{xbx\}$ is nilpotent, there exists a k such that $y = \{xbx\}^{2^{k-1}} \notin \mathfrak{C}$ but $y^2 = \{xbx\}^{2^k} \in \mathfrak{B}$. Moreover, y has the form $\{xb^*x\}$, where $b^* \in \mathfrak{B}$, as can be seen by induction on k using identity (23). We have $yb' \in \mathfrak{C}$ for every $b' \in \mathfrak{B}$. We assert that $yb' \in \mathfrak{B}$, and, since $y^2 \in \mathfrak{C}$, this will imply that y satisfies the conditions on x in the statement of the lemma. Thus let $\mathfrak{B}_1$ be the subspace of $\mathfrak{B}$ of elements b_1 such that $yb_1 \in \mathfrak{B}$. By Lemma 3, $\mathfrak{B}_1 \supseteq \mathfrak{B}^2$. Since $y^2 \in \mathfrak{B}$, $\mathfrak{C} + \Phi y$ is a subalgebra of $\mathfrak{A}$. Let $b_1 \in \mathfrak{B}_1$, and consider $y(yb_1) = \frac{1}{2}y^2b_1 + \{yb_1y\}$. Since $y^2 \in \mathfrak{B}$, $y^2b_1 \in \mathfrak{B}^2 \subseteq \mathfrak{B}_1$. Hence $\{yb_1y\} \in \mathfrak{C}$, but, since it is not regular, $\{yb_1y\} \in \mathfrak{B}$. Hence $y(yb_1) \in \mathfrak{B}$ and $yb_1 \in \mathfrak{B}_1$, by definition of $\mathfrak{B}_1$. This implies that $\mathfrak{B}_1$ is an ideal in $\mathfrak{C} + \Phi y$. Since $y^2 \in \mathfrak{B}$, $y^3 \in \mathfrak{C}$; but, being nilpotent, $y^3 \in \mathfrak{B}$. Hence $y^2 \in \mathfrak{B}_1$. Suppose that $\mathfrak{B} \neq \mathfrak{B}_1$, and let $b \in \mathfrak{B}$, $\notin \mathfrak{B}_1$. Then, by replacing b by a suitable multiple, we may suppose that $2yb = u + b'$. Hence $yb' = 2y(yb) - y = y^2b + 2\{yby\} - y$. We wish to show that this is in $\mathfrak{B}$. Since $y^2b \in \mathfrak{B}$, we have to show that $2\{yby\} - y \in \mathfrak{B}$. We square and obtain $4\{yby\}^2 + y^2 - 4y\{yby\} = 4\{y\{by^2b\}y\} + y^2 - 4y\{yby\}$. Since $y^2 \in \mathfrak{B}$, $\{by^2b\} \in \mathfrak{B}^2$, and, by Lemma 3, $\{y\{by^2b\}y\} \in \mathfrak{B}$. Also, $y\{yby\} = bR_y^3 - \frac{1}{2}(by^2)y$, and since $y^2 \in \mathfrak{B}_1$, $bR_y^3 = \frac{3}{2}bR_y^2R_y - \frac{1}{2}bR_y^3 \in \mathfrak{B}$. Hence $y\{yby\} \in \mathfrak{B}$,

and so $(2\{yby\} - y)^2 \in \mathfrak{B}$. Since $2\{yby\} - y = yb' - y^2b \in \mathfrak{C}$, this implies that $2\{yby\} - y \in \mathfrak{B}$. Hence, also, $yb' \in \mathfrak{B}$, and so $b' \in \mathfrak{B}_1$. Hence $2yb \equiv u \pmod{\mathfrak{B}_1}$, and

$$\left(\frac{1}{2}(u + y + b)\right)^2 \equiv \frac{1}{4}(u + 2(y + b) + 2yb)$$

$$\equiv \frac{1}{2}(u + y + b) \pmod{\mathfrak{B}_1}.$$

Set $f = \frac{1}{2}(u + y + b)$, and let $\mathfrak{D}$ be the subalgebra generated by f. This algebra is associative and has the nil ideal $\mathfrak{D} \cap \mathfrak{B}_1$. The coset of f in $\mathfrak{D}/(\mathfrak{D} \cap \mathfrak{B}_1)$ is idempotent. Hence we may choose an idempotent g in this coset. Now it is clear that u and 0 are the only idempotent elements of $\mathfrak{A}$. Hence $g = u$ or $g = 0$. In the first case, $u \equiv \frac{1}{2}(u + y + b) \pmod{\mathfrak{B}_1}$, and, in the second, $0 \equiv \frac{1}{2}(u + y + b) \pmod{\mathfrak{B}_1}$. Either of these implies that $y \in \mathfrak{C}$, contrary to assumption. We have therefore proved that $yb \in \mathfrak{B}$ for every $b \in \mathfrak{B}$, and y satisfies our conditions.

We may now prove Theorem A. Thus, let $\mathfrak{C}$ be a maximal subalgebra of $\mathfrak{A}$ of the form $\Phi u + \mathfrak{B}$, where $\mathfrak{B}$ is a nil subalgebra. If $\mathfrak{C} \neq \mathfrak{A}$, we can find a nilpotent element $x \notin \mathfrak{C}$ such that xb and $x^2b \in \mathfrak{B}$ for every $b \in \mathfrak{B}$. Then $\mathfrak{C} + \Phi x$ is a subalgebra properly containing $\mathfrak{C}$, and $\mathfrak{B}$ is an ideal in $\mathfrak{B} + \Phi x$ whose difference algebra is a nil algebra. Hence $\mathfrak{B} + \Phi x$ is a nil algebra, and $\mathfrak{C} + \Phi x = \Phi u + (\mathfrak{B} + \Phi x)$. This contradicts the maximality of $\mathfrak{C}$. Hence $\mathfrak{A} = \mathfrak{C} = \Phi u + \mathfrak{B}$.

As we noted before, Theorem A implies that the only finite-dimensional simple Jordan algebra over an algebraically closed field having only one idempotent $\neq 0$ is $\mathfrak{A} = \Phi u$. This and our earlier results imply that the representations of finite-dimensional separable Jordan algebras are all completely reducible.[6] Also, one can determine the irreducible representations using the structure of $\mathfrak{A}$.

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[1] A. A. Albert, "On Jordan Algebras of Linear Transformations," *Trans. Am. Math. Soc.*, **59**, 524–555, 1946; "A Structure Theory for Jordan Algebras," *Ann. Math.*, **48**, 546–567, 1947; "A Theory of Power-associative Commutative Algebras," *Trans. Am. Math. Soc.*, **69**, 503–527, 1950.

[2] It is shown by Albert, in "A Theory of Power-associative Commutative Algebras," p. 523, that every finite-dimensional simple Jordan algebra has an identity.

[3] See the author's "General Representation Theory of Jordan Algebras," *Trans. Am. Math. Soc.*, **70**, 509–530, 1951; "Structure of Alternative and Jordan Bimodules," *Osaka Math. J.*, **6**, 1–71, 1954.

[4] All of these except equation (6) can be found in Albert's, "A Structure Theory for Jordan Algebras." Equation (6) is due to W. H. Mills, "A Theorem on the Representation of Jordan Algebras," *Pacific J. Math.*, **1**, 255–264, 1951. The form (2′) of relation (2) is due to R. D. Schafer, to whom I am also indebted for simplifications in the proofs of identities (19)–(22) below.

[5] Albert, "A Structure Theory for Jordan Algebras," Theorems 1 and 3. The proof which we give of Theorem A is an adaptation of Albert's, given in "A Theory of Power-associative Commutative Algebras," pp. 514–516.

[6] See N. Jacobson, "Structure of Alternative and Jordan Bimodules," p. 69.

ON INVARIANT SKEW-TENSORS

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Throughout this note V will denote a real n-dimensional vector space with a positive-definite inner product, its value for $u, v \in V$ being denoted by (u, v). Let $\wedge^p V$ be the space of contravariant skew-tensors over V of degree p. The inner product in V induces a positive-definite inner product in $\wedge^p V$, where $(u_1 \wedge \ldots \wedge u_p, v_1 \wedge \ldots \wedge v_p) = (1/p!) \cdot \text{determinant } (u_i, v_j)$.

Let $\mathfrak{k}$ be a Lie algebra of a compact group. A representation of $\mathfrak{k}$ on V will be denoted by the triple $\{\sigma, \mathfrak{k}, V\}$. It will be always understood that the representation is by skew-symmetric operators (skew-symmetric with respect to the given inner product), the operator corresponding to $X \in \mathfrak{k}$ being denoted by $\sigma(X)$.

Given $\{\sigma, \mathfrak{k}, V\}$, we will let $\{\sigma^p, \mathfrak{k}, \wedge^p V\}$ be the corresponding representation of $\mathfrak{k}$ on $\wedge^p V$, where, as one knows, $\sigma^p(X) v_1 \wedge \ldots \wedge v_p = \sum_{i=1}^{p} v_1 \wedge \ldots \wedge \sigma(X) v_i \wedge \ldots \wedge v_p$. Note that $\{\sigma^0, \mathfrak{k}, \wedge^0 V\}$ is the zero representation of $\mathfrak{k}$.

Now let $W_p = \{s \in \wedge^p V \mid \sigma^p(X) s = 0 \text{ for all } X \in \mathfrak{k}\}$. The elements of W_p are called the pth-order skew-invariants of $\{\sigma, \mathfrak{k}, V\}$. Let d_p be the dimension of W_p. We will call $P_\sigma(t) = \sum_{p=0}^{n} d_p t^p$ the *Poincaré polynomial* of $\{\sigma, \mathfrak{k}, V\}$.

We consider for the present the special class of representations which one associates to compact homogeneous Riemannian symmetric manifolds (hereafter called *symmetric representations*). That is, let $M = G/K$ be such a manifold, G being the connected component of the identity in the full group of motions of M. We consider the representation $\{\sigma, \mathfrak{k}, V\}$, where V is the tangent space at the point of M corresponding to the coset K (with the inner product given by the metric tensor), $\mathfrak{k}$ is the Lie algebra of K, and σ is the representation of $\mathfrak{k}$ on V induced by the obvious action of K on V. All symmetric representations have been studied and classified by É. Cartan. He has shown that they are all of the form ad $\mathfrak{k}$ restricted to V (ad $\mathfrak{k} \mid V$), where $\mathfrak{g} = \mathfrak{k} + V$ is a Lie algebra satisfying (*a*) $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$, (*b*) $[V, V] = \mathfrak{k}$, and (*c*) $[\mathfrak{k}, V] \subseteq V$, where ad $\mathfrak{k} \mid V$ is faithful and skew-symmetric with respect to a positive-definite inner product on V. Conversely, every such representation is a symmetric representation. Moreover, M may be chosen so that K is connected and hence as one knows the Poincaré polynomial of M is precisely $P_\sigma(t)$.

We shall call $\{\sigma, \mathfrak{k}, V\}$ an S-representation if it is irreducible and symmetric. One knows all symmetric representations when one knows the S-representations.[1]

The S-representations form a fairly extensive class (they are, of course, all faithful). Important for our purposes is a "small" subclass whose elements we call *T-representations*. We say that $\{\sigma, \mathfrak{k}, V\}$ is a T-representation if it is an S-representation and $\mathfrak{k}$ is simple (it may be the zero- or one-dimensional Lie algebra). These representations will be characterized among *all* representations by Theorem 1.

The T-representations are of two types: T_1, the adjoint representations of the simple (non-Abelian) compact Lie algebras, and T_2, the nonadjoint representations. The latter are listed below; they are specified by giving a corresponding compact

homogeneous Riemannian symmetric space: Type T_2: (*a*) $SU(m)/SO(m)$, $m = 3, 5, \ldots$; (*b*) $SU(2m)/Sp(m)$, $m = 3, \ldots$; (*c*) $SO(m+1)/SO(m)$, $m = 1, 2, 5, \ldots$; (*d*) $F_4/Spin$ (9); (*e*) E_6/F_4; (*f*) $E_6/\bar{S}p$ (4); (*g*) $E_7/S\bar{U}$ (8); (*h*) $E_8/Spin(16)$.[2]

The representations $\{\sigma, \mathfrak{k}, V\}$ corresponding to $SO(n+1)/SO(n)$, $n = 1, \ldots$, we shall call *complete*. They are complete in the sense that $\mathfrak{k}$ is isomorphic to the Lie algebra of all skew-symmetric operators on V, and σ gives the isomorphism. That is, $\{\sigma, \mathfrak{k}, V\}$ is the "natural" representation of $\mathfrak{k}$. Note that they are all S-representations, and in fact, except for $n = 3, 4$, all are T_2-representations. When $n = 4$, $\{\sigma, \mathfrak{k}, V\}$ is non-T. We shall call the complete representation the *trivial* representation[1] when $n = 1$ and the *circular* representation when $n = 2$.

Now let $\{\sigma, \mathfrak{k}, V\}$ be an arbitrary representation. It is not difficult to give necessary and sufficient conditions that $d_2 = 0$. Our first main result hinges upon d_4. We wish to point out that the proof, outlined below together with that of Theorem 1′, does not employ the classification (but only the definition) of T-representations. We have

THEOREM 1. *Let $\{\sigma, \mathfrak{k}, V\}$ be any faithful irreducible representation. Then $d_4 = 0$ if and only if it is a T-representation.*

Remark 1; Assuming the condition of Theorem 1 ($\{\sigma, \mathfrak{k}, V\}$ is a T-representation), one has (*a*) $d_3 = 0$ if and only if it is type T_2, (*b*) $d_2 = 0$ if and only if it is non-circular, and (*c*) $d_1 = 0$ if and only if it is nontrivial.

Given the arbitrary representations $\{\sigma_i, \mathfrak{k}_i, V_i\}$, $i = 1, 2, \ldots, q$, we define the composite to be the representation $\left\{\sigma, \sum_{i=1}^{q} \mathfrak{k}_i, \sum_{i=1}^{q} V_i\right\}$ defined by setting $\sigma\left(\sum_{i=1}^{q} X^i\right)\sum_{i=1}^{q} v^i = \sum_{i=1}^{q} \sigma_i(X^i)v_i$, where $X^i \in \mathfrak{k}_i$, $v^i \in V_i$. One can show without difficulty that $P_\sigma(t) = \prod_{i=1}^{q} P_{\sigma_i}(t)$.

We now have, more generally,

THEOREM 1′. *Let $\{\sigma, \mathfrak{k}, V\}$ be an arbitrary faithful representation. Then $d_4 = 0$ if and only if it is a composite of T-representations $\{\sigma_i, \mathfrak{k}_i, V_i\}$, $i = 1, 2, \ldots, q$, such that the coefficient of t^4 in $\prod_{i=1}^{q} P_{\sigma_i}(t)$ is zero.*

Remark 2: The determination of the coefficient of t^4 in the above theorem clearly involves only a knowledge of the coefficients of 1, t, t^2, and t^3 in the factors. But from the remark following Theorem 1, in order that this coefficient vanish, one has only to avoid certain obvious combinations of the trivial, circular, and T_1-representations.

Proof (outline) of Theorem 1: Assume that an arbitrary $\{\sigma, \mathfrak{k}, V\}$ is given. We make two permissible identifications: (1) $\wedge^2 V$ is identified with the Lie algebra of all skew-symmetric operators on V, where $u_1 \wedge u_2 \cdot v = {}^1/_2(u_1, v)u_2 - {}^1/_2(u_2, v)u_1$ (using a dot for the operation on vectors $v \in V$); (2) $\wedge^4 V$ is identified with all operators S on $\wedge^2 V$ which satisfy $S(u_1 \wedge u_2)\cdot u_3 \in V$ is skew-symmetric in $u_1, u_2, u_3 \in V$. In fact, S is that element of $\wedge^4 V$ which satisfies $(S, u_1 \wedge u_2 \wedge u_3 \wedge u_4) = (S(u_1 \wedge u_2)\cdot u_4, u_3)$. Now it is not difficult to show that if $X \in \mathfrak{k}$, $A \in \wedge^2 V$, $S \in \wedge^4 V$, $\sigma_2(X)A = [\sigma(X), A] = \operatorname{ad} \sigma(X)A$ and $\sigma_4(X)S = [\operatorname{ad} \sigma(X), S]$.

Now let R: $\wedge^2 V \to \sigma(\mathfrak{k})$ be the orthogonal projection on the subalgebra $\sigma(\mathfrak{k}) \subseteq \wedge^2 V$. One may verify that, since R is a symmetric operator, $S(u_1 \wedge u_2)\cdot u_3 =$

$1/3\ (R(u_1 \wedge u_2)\cdot u_3 + R(u_2 \wedge u_3)\cdot u_1 + R(u_3 \wedge u_1)\cdot u_2)$ defines an operator S on $\wedge^2 V$. Clearly $S(u_1 \wedge u_2)\cdot u_3$ is skew-symmetric in $u_1, u_2, u_3 \in V$. Thus $S \in \wedge^4 V$. But now observe that R commutes with ad $\sigma(X)$ for $X \in \mathfrak{k}$. It follows, then, that S also commutes with all ad $\sigma(X)$. That is, $[\text{ad } \sigma(X), S] = 0$, or, from the above, $\sigma_4(X)S = 0$, and hence $S \in W_4$.

Now form the vector space $\mathfrak{g} = \mathfrak{k} + V$. Define a bracket relation in $\mathfrak{g}$ by (1) setting $[X, v] = -[v, X] = \sigma(X)v \in V$; (2) also $[v_1, v_2] = o^{-1} R(v_1, v_2)$ for $v, v_1, v_2 \in V$, and $X \in \mathfrak{k}$; and (3) retaining the original bracket relation in $\mathfrak{k}$. Assume now that $d_4 = 0$. Then we must have $S = 0$. But $S = 0$ is exactly the condition that the bracket relation defined in $\mathfrak{g}$ satisfy the Jacobi relation, and hence $\mathfrak{g}$ is a Lie algebra. It follows that, since conditions (a), (b), and (c) of the fifth paragraph, above, are satisfied, $\{\sigma, \mathfrak{k}, V\}$ is a symmetric representation. Now it is known (due to É. Cartan) that any symmetric representation is a composite of S-representations $\{\sigma_i, \mathfrak{k}_i, V_i\}$, $i = 1, \ldots, q$, and hence $P_\sigma(t) = \prod_{i=0}^{q} P_{\sigma_i}(t)$. Thus, to prove the theorem, it suffices to show that for an S-representation, $\{\sigma, \mathfrak{k}, V\}$, $d_4 = 0$ if and only if $\mathfrak{k}$ is simple. That is, if G/K is the corresponding compact homogeneous Riemannian symmetric space, the fourth Betti number is zero if and only if the Lie algebra, $\mathfrak{k}$, of K is simple. This latter fact may be proved by using spectral sequences. (If $\mathfrak{k}$ is not Abelian, the case is easily reduced to that in which $\mathfrak{k}$ is semisimple and $\mathfrak{g}$ is simple. The injection $\mathfrak{k} \to \mathfrak{g}$ then carries the third real cohomology group of $\mathfrak{g}$ onto the third real cohomology group of $\mathfrak{k}$ when and only when $\mathfrak{k}$ is simple.) *Q.E.D.*

Now it is well known that the Poincaré polynomial $P_\sigma(t)$ of a complete representation $\{\sigma, \mathfrak{k}, V\}$ is $1 + t^n$. That is, there are no nontrivial skew-invariants. It is natural to ask which other representations have this property. The answer is given in

THEOREM 2. *Let the Poincare polynomial $P_\sigma(t)$ of the faithful representation $\{\sigma, \mathfrak{k}, V\}$ be $1 + t^n$ (where $n = \dim V$). Then either $\{\sigma, \mathfrak{k}, V\}$ is complete or $\{\sigma, \mathfrak{k}, V\}$ corresponds to $SU(3)/SO(3)$ (type T_2 (a), for $m = 3$).*

Proof: Assume $P_\sigma(t) = 1 + t^n$. It follows immediately that $\{\sigma, \mathfrak{k}, V\}$ must be irreducible, since, if there is an invariant subspace of dimension m, then $d_m \neq 0$.

Now assume $n \neq 4$. Then, by Theorem 1, $\{\sigma, \mathfrak{k}, V\}$ is a T-representation. We eliminate every T_1-representation except $SO(4)/SO(3)$, which is complete, since $d_3 \neq 0$ for the T_1-representations. We may eliminate cases (g) and (h) of the T_2-representations, since the Euler characteristic of the corresponding compact symmetric space is greater than 2. We eliminate cases (d), (e), and (f), since the Euler characteristic of the corresponding space is zero but the spaces are even-dimensional. We eliminate case (a), $m \neq 3$, and case (b), since in all these cases (including case (a), $m = 3$) the fifth Betti number of the corresponding compact symmetric spaces is not zero (see Borel,[3] pp. 203, 204), and hence $d_5 \neq 0$. Also using the foregoing reference, we see that for type T_2, (a), $m = 3$, $P_\sigma(t) = 1 + t^5$.

Assume $n = 4$. Now it is clear that the center of $\mathfrak{k}$ is zero, since otherwise $d_2 \neq 0$. But if $\{\sigma, \mathfrak{k}, V\}$ is not complete, $\mathfrak{k}$ can only be isomorphic to the Lie algebra of $SO(3)$, and there exists only one irreducible representation of this algebra on a 4-dimensional space. But, for this representation, $d_2 \neq 0$. Hence $\{\sigma, \mathfrak{k}, V\}$ is complete. *Q.E.D.*

Applications: Let M be any simply connected Riemannian manifold (or, more generally, any Riemannian manifold whose holonomy group is connected). Let V be the tangent space at a point $o \in M$ (with the inner product given by the metric tensor). Let $\mathfrak{k}$ be the Lie algebra of the holonomy group at o, and $\{\sigma, \mathfrak{k}, V\}$ the obvious representation of $\mathfrak{k}$ on V. We shall call $\{\sigma, \mathfrak{k}, V\}$ the holonomy representation associated to M.

Now one knows that there is a natural one-to-one relation between the skew-invariants of $\{\sigma, \mathfrak{k}, V\}$ and all covariant constant differential forms Ω on M ($\nabla_v \Omega = 0$, where ∇_v is covariant differentiation by any tangent vector v on M). For simplicity assume that M is irreducible ($\{\sigma, \mathfrak{k}, V\}$ is irreducible). To say that $P_\sigma(t) \neq 1 + t^n$ is a very strong assumption. For example, $d_2 \neq 0$ if and only if M is Kählerian. In fact, Lichnerowicz has conjectured that if $P_\sigma(t) \neq 1 + t^n$, then M is either Kähler or symmetric. If this conjecture is true, it follows from Theorem 2 that for every (say) simply connected, irreducible, non-Kähler, nonsymmetric manifold, when $n \neq 5$, $\{\sigma, \mathfrak{k}, V\}$ is complete! Even for a 5-dimensional manifold the exceptional case of Theorem 2 cannot occur, since otherwise the holonomy group would not be transitive on the unit sphere of V. This contradicts a result of Berger.

Now, observing that every covariant constant form is harmonic, we have

COROLLARY 2.1. *Let M be a simply connected, irreducible Riemannian manifold which admits no covariant constant differential 4-form (the latter holding, for example, if M is compact and the fourth Betti number is zero). Then the holonomy representation $\{\sigma, \mathfrak{k}, V\}$ is (a) a T-representation. Moreover (b), if M is not Riemannian symmetric, $\{\sigma, \mathfrak{k}, V\}$ is actually complete unless dim $M = 16$, in which case $\{\sigma, \mathfrak{k}, V\}$ either is complete or corresponds to $F_4/Spin$ (9).*

Proof: Part (*a*) follows from Theorem 1. Part (*b*) follows from the results of Berger, since, besides the complete representations, $F_4/Spin$ (9) is the only T-representation for which the corresponding group representation is transitive on the unit sphere of V. *Q.E.D.*

Remark 3: Even $F_4/Spin(9)$ cannot occur in the above theorem if there exists no covariant constant 8-form on M.

As a final application, we generalize (except for $n = 5$) the theorem of Iwamoto on the holonomy group of the spheres, S_n, with an arbitrary Riemann metric. (Recently, also proved by Singer for even-dimensional spheres.) We have

COROLLARY 2.2. *Let M be a compact, simply connected Riemannian manifold which is, topologically, a rational homology sphere. Then, except when $n = 5$, the holonomy representation $\{\sigma, \mathfrak{k}, V\}$ is always complete. When $n = 5$, $\{\sigma, \mathfrak{k}, V\}$, besides being complete, may correspond to $SU(3)/SO(3)$.*

Proof: This follows immediately from Theorem 2 and, of course, from the Hodge theorem.

Remark 4: Notice that, using the results of Berger, the exceptional case occurs when and only when M is locally isometric to $SU(3)/SO(3)$.

[1] Note that the representation of the zero-dimensional Lie algebra on a one-dimensional vector space is an S-representation.

[2] F_4, E_6, E_7, and E_8 are taken as the corresponding simply connected compact groups. $\tilde{S}p(4)$ means $\tilde{S}p(4)$ modulo its center, and $\tilde{SU}(8)$ means $SU(8)$ modulo the 2-element subgroup $\{1, -1\}$.

[3] A. Borel, "Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts," *Ann. Math.*, **57**, 115–207, 1953.

REMARK ON KINEMATICALLY PREFERRED CO-ORDINATE SYSTEMS

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In defining *kinematically preferred co-ordinate systems,*[1] it is observed that the determination of these systems depends essentially on the velocity field within the medium. Specifically, this dependence occurs through the behavior of the curl of the velocity field and an initial condition prescribed for the differential equations at a time $t = t_0$ by which we determine the orthogonal coefficients relating the kinematically preferred systems y to the co-ordinate systems x with which the y systems are associated.

The purpose of the present note is to show that *kinematically preferred co-ordinate systems with origins at the same moving point $P(t)$ are completely determined up to an orthogonal transformation with constant coefficients.* More precisely, we wish to show that if we prescribe the same[2] initial condition at some time $t_0^* \neq t_0$, obtaining thus a kinematically preferred system y^*, then for all times t the systems y and y^* are related by equations of the form

$$y_\alpha = e_{\alpha\beta} y_\beta^*, \tag{1}$$

where the coefficients $e_{\alpha\beta}$ are orthogonal and do not depend on t.

The co-ordinate systems y and y^* are related to a fixed co-ordinate system x by means of the equations

$$x_\alpha = c_{\alpha\beta}(t) y_\beta + g_\alpha(t), \tag{2}$$

$$x_\alpha = c_{\alpha\beta}^*(t) y_\beta^* + g_\alpha(t), \tag{3}$$

where $g_\alpha(t)$ is the trajectory of a moving point $P(t)$ of the medium (point at which the y and y^* systems have their common origins, see Fig. 1) and where the coefficients $c_{\alpha\beta}(t)$ and $c_{\alpha\beta}^*(t)$ are orthogonal, satisfying furthermore the linear differential equations with nonconstant coefficients

$$\dot{c}_{\alpha\beta} = c_{\sigma\beta}\phi_{\alpha\sigma}(t); \qquad c_{\alpha\beta}(t_0) = \delta_{\alpha\beta}, \tag{4}$$

$$\dot{c}_{\alpha\beta}^* = c_{\sigma\beta}^*\phi_{\alpha\sigma}(t); \qquad c_{\alpha\beta}^*(t_0^*) = \delta_{\alpha\beta}, \tag{5}$$

in which $\phi_{\alpha\beta}(t)$ denotes the components of the curl of the velocity field of the motion under consideration relative to the x system.

We may now prescribe $c_{\alpha\beta}(t_0^*)$ as a new initial value for equation (4) at the time $t = t_0^*$; and, if we denote by $c_{\alpha\beta}^*(t)$ the solution of equation (5), then, from the uniqueness theorem, the solution of equation (4) with its new initial data is given by

$$c_{\alpha\epsilon}(t) = c_{\beta\epsilon}(t_0^*) c_{\alpha\beta}^*(t). \tag{6}$$

Comparison of equations (2) and (3), and the orthogonality of the coefficients $c_{\alpha\beta}$ and $c_{\alpha\beta}^*$, show that, together with equation (6), we have

$$\begin{aligned} y_\epsilon &= c_{\alpha\epsilon}(t)c^*_{\alpha\sigma}(t)y^*_\sigma \\ &= c_{\beta\epsilon}(t^*_0)c^*_{\alpha\beta}(t)c^*_{\alpha\sigma}(t)y^*_\sigma \\ &= c_{\beta\epsilon}(t^*_0)\delta_{\beta\sigma}y^*_\sigma \\ &= c_{\beta\epsilon}(t^*_0)y^*_\beta, \end{aligned}$$

which is the desired result.

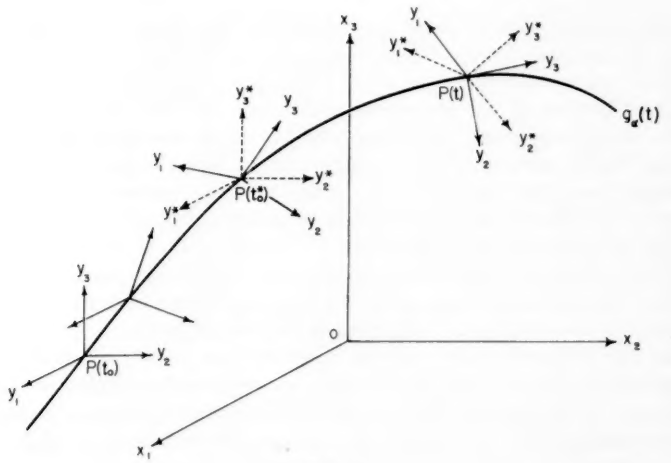

FIG. 1

It may be of some interest to observe that a converse result is also true. More precisely, *given a kinematically preferred co-ordinate system y and any other rectangular system y^* related to y by means of equations* (1), *then y^* is also a kinematically preferred co-ordinate system.* To see this, it suffices to combine equations (1) and (2) in order to obtain equation (3), in which we have now

$$c^*_{\alpha\beta}(t) = c_{\alpha\epsilon}(t)e_{\epsilon\beta}.$$

The differential equations (5) will then still be valid, but the initial condition will now read $c^*_{\alpha\beta}(t_0) = e_{\alpha\beta}$. If one wishes to normalize this initial condition, one may introduce a new co-ordinate system $\bar{x}$ by the transformation $x_\alpha = e_{\alpha\beta}\bar{x}_\beta$. This transformation will in turn induce a transformation upon the kinematically preferred system y, yielding a new system $\bar{y}$. It can then be immediately verified that, as one would expect, the $\bar{y}$ and y^* systems will coincide.

[1] See T. Y. Thomas, "Kinematically Preferred Co-ordinate Systems," these PROCEEDINGS, **41**, 762–770, 1955.

[2] That any *other* initial condition would still yield our result is immediately obvious from the proof given here. It is desirable, however, to normalize the initial condition (i.e., reduce it to the unit matrix) in order to simplify later formulas.

*ON CONSISTENT ESTIMATES OF THE SPECTRAL DENSITY OF A STATIONARY TIME SERIES**

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1. *Introduction.*—The problem of estimating the power spectrum of noise (that is, the spectral density function of a stationary time series) has attracted much attention recently[1] because of its increasing importance in many scientific studies and because of the increasing realization that the widely used obvious estimate, the periodogram, is not consistent. Several consistent estimates have been suggested by various authors, among whom one should name Bartlett, Daniell, Grenander, Rosenblatt, and Tukey.[1] In this note we consider a general class of estimates of the spectral density function and obtain expressions for their asymptotic bias, variance, and mean-square error. Proofs of these results, as well as an elaboration of the theory presented here, will be published elsewhere. It is believed that these results provide the means of choosing estimates which may be said to be optimum, but this point is not discussed in the present note.

To say that a function $x(t)$ is a time series or a stochastic process or a random function is to say that the function $x(t)$ does not have a definite value for each value of t but only an ensemble of possible values which it assumes in accordance with some given probability distribution, an average with respect to which is denoted by the expectation operator E. The parameter t may be continuous ($-\infty < t < \infty$) or discrete ($t = 0, \pm 1, \ldots$). A time series is said to be stationary (in the wide sense) if its mean value $Ex(t)$ is equal to a constant, denoted by m, for all t, and its product moment $Ey(t)\ y(t+v)$, where $y(t) = x(t) - m$, is a function only of v, denoted by $R(v)$ and called the *covariance function*.

In this note it is assumed that the covariance function is both absolutely summable and square summable. Then there is a continuous non-negative function $f(w)$, called the *spectral density function* of the time series, satisfying the relation

$$R(v) = \int e^{ivw} f(w)\, dw, \tag{1.1}$$

where the domain of $f(w)$, and the limits of integration in equation (1.1), are $-\infty$ to ∞ in the continuous-parameter case and $-\pi$ to π in the discrete-parameter case.

In this note it is also assumed that $y(t)$ is stationary of order 4, in the sense that the fourth moment $Ey(t)\ y(t+v_1)\ y(t+v_2)\ y(t+v_3)$ is a function only of the time differences v_1, v_2, and v_3, which we denote by $P(v_1, v_2, v_3)$. Now if $x(t)$ were a Gaussian process, then $P(v_1, v_2, v_3)$ could be expressed in terms of the covariance function $R(v)$ by

$$P_G(v_1, v_2, v_3) = R(v_1)\, R(v_2 - v_3) + R(v_2)\, R(v_3 - v_1) + R(v_3)\, R(v_1 - v_2).$$

We define the "non-Gaussian" part of the fourth moment by

$$P_{NG}(v_1, v_2, v_3) = P(v_1, v_2, v_3) - P_G(v_1, v_2, v_3).$$

We assume that $P_{NG}(v_1, v_2, v_3)$ is absolutely summable.

2. *The Problem of Empirical Spectral Analysis.*—We discuss only the case of discrete-parameter time series. The case of continuous-parameter time series may be discussed similarly. Also, in order not to burden the argument, in the present note we assume that the mean value m vanishes, although all the results stated may be extended to the case where the mean value has to be estimated from the sample.

Let the time series $x(t)$ be observed for $t = 1, \ldots, T$. The problem of empirical spectral analysis of stationary time series is to use these observations to form a consistent estimate of the spectral density function. An estimate which has been considered is the periodogram, defined by

$$f_T(w) = \frac{1}{T}\left|\sum_{t=1}^{T} e^{-iwt}\, x(t)\right|^2$$

$$= \frac{1}{2\pi}\sum_{|v|\leq T} e^{-ivw}\, R_T(v),$$

where, for $|v| \leq T$,

$$R_T(v) = \frac{1}{T}\sum_{t=1}^{T-|v|} x(t)\, x(t + |v|).$$

However, various authors[1] have pointed out that the periodogram is not a consistent estimate of the spectral density. The suggestion has been made to estimate $f(w)$ at a point w_0 by averaging the values of $f_T(w)$ in the neighborhood of w_0. However, this yields a consistent estimate not of $f(w_0)$ but rather of an average over the spectral density in the neighborhood of w_0. To eliminate this bias, one needs to narrow the neighborhood averaged over as the sample size is increased. Another method of obtaining a consistent estimate of spectral density is that of Bartlett, who has suggested a modified form of the periodogram. These methods are generalized in this paper.

3. *Averaging Kernels.*—A function $k(z)$, defined for all real z, will be called a (*covariance*) *averaging kernel* if it is even, bounded, square integrable, and normalized so that $k(0) = 1$ and if it satisfies the condition

$$\int_{-T}^{T} |k(z)|\, dz \leq MT^{(1/2-\epsilon)} \tag{3.1}$$

for some constants M and $\epsilon > 0$. A kernel is said to be of class K_r, where $r > 0$, if r is the largest positive number such that

$$k^{(r)} = \lim_{z\to 0} \frac{1 - k(z)}{|z|^r} \tag{3.2}$$

exists and is finite. An important example of an averaging kernel of class K_r is

$$\begin{aligned} k_r(z) &= 1 - |z|^r \quad \text{for } |z| < 1, \\ &= 0 \quad\quad\quad\;\; \text{for } |z| > 1. \end{aligned} \tag{3.3}$$

4. *Estimates of the Spectral Density.*—Let $k(z)$ be an averaging kernel, and let B_T be a sequence of positive constants tending to 0 as $T \to \infty$ in such a way that $B_T T \to \infty$. Define

$$f_T^*(w) = \frac{1}{2\pi} \sum_{|v| \leq T} e^{-ivw} k(B_T v) R_T(v). \tag{4.1}$$

Various estimates of the spectral density which have been proposed[1] may be obtained as instances of equation (4.1). By letting $k(z) = k_1(z)$ and $B_T = 1/M$, where M is an integer less than T, one obtains Bartlett's estimate:

$$f_B(w) = \frac{1}{2\pi} \sum_{v=-M}^{M} e^{-ivw} \left(1 - \frac{|v|}{M}\right) R_T(v).$$

Similarly, one may obtain Daniell's estimate by letting $k(z) = (\sin z)/z$ and $B_T = h$, and the truncated estimate by letting $k(z) = k_\infty(z)$ and $B_T = 1/M$. This last estimate has the possibly undesirable property that it is not necessarily non-negative.

5. *Properties of the Estimates.*—The properties of the estimate $f_T^*(w)$ are embodied in the following two theorems.

THEOREM A. *The covariance of the estimate $f_T^*(w)$ satisfies, for non-negative w_1 and w_2,*

$$\lim_{T\to\infty} TB_T \, Cov\,[f_T^*(w_1), f_T^*(w_2)] = \int k^2(z)\, dz \, |f(w_1)|^2 \{1 + \delta(0, w_1)\} \, \delta(w_1, w_2), \tag{5.1}$$

where $\delta(w_1, w_2) = 1$ or 0 according as $w_1 = $ or $\neq w_2$. Further, for any $\epsilon > 0$, the limit in equation (5.1) is uniform in w_1 and w_2 such that $w_1 > \epsilon$ and $w_2 > \epsilon$.

THEOREM B. *Let $r > 0$ be such that*

$$\sum |v|^r \, |R(v)| < \infty. \tag{5.2}$$

Let $k(z)$ be an averaging kernel of class K_r. Then the bias of the estimate $f_T^*(w)$ satisfies

$$\lim_{T\to\infty} B_T^{-2r} |Ef_T^*(w) - f(w)|^2 = |k^{(r)} f^{(r)}(w)|^2 \tag{5.3}$$

uniformly in w, where

$$f^{(r)}(w) = \frac{1}{2\pi} \sum e^{-ivw} |v|^r R(v). \tag{5.4}$$

6. *Order of Consistency.*—The basic requirement for an estimate $f_T^*(w)$ of the spectral density $f(w)$ is that it be consistent in quadratic mean; that is, $E|f_T^*(w) - f(w)|^2 \to 0$ as $T \to \infty$. The estimate is then asymptotically unbiased, which means that $Ef_T^*(w) \to f(w)$ as $T \to \infty$. However, the rate of consistency and unbiasedness is also important. Let α be a positive number. We define an estimate to be asymptotically unbiased of the order of T^α if

$$\lim_{T\to\infty} T^\alpha |Ef_T^*(w) - f(w)| = \beta < \infty \tag{6.1}$$

and to possess an asymptotic variance σ^2 of the order of $T^{2\alpha}$ if $\sigma > 0$ and

$$\lim_{T\to\infty} T^{2\alpha} E|f_T^*(w) - Ef_T^*(w)|^2 = \sigma^2. \tag{6.2}$$

If equations (6.1) and (6.2) hold, we define $f_T^*(w)$ to be consistent of the order of $T^{2\alpha}$, with asymptotic mean-square error $\sigma^2 + \beta^2$.

A knowledge of the order of consistency and asymptotic mean-square error of an estimate is valuable on several counts. First, it may be used to choose between competing consistent estimates of the same quantity. Second, it is useful for the design of the analysis of observed time series, by giving estimates of the sampling variability arising from the length of observation. Third, it may be used to construct asymptotic confidence bands for the quantity being estimated (if the central limit theorem applies).

We next state a theorem which shows that from the general class of estimates $f_T^*(w)$ it is possible to select estimates which are consistent of any prescribed order $T^{2\alpha}$, where $0 < \alpha < 1/2$.

THEOREM C. *Suppose that relation* (5.2) *holds. Let the constants* B_T *be chosen to satisfy*

$$\lim_{T\to\infty} T^{1/(1+2r)} B_T = B > 0. \tag{6.3}$$

Let

$$\alpha = \frac{r}{1+2r}. \tag{6.4}$$

Then, for any averaging kernel of class K_r, *the corresponding estimate* $f_T^*(w)$ *satisfies*

$$\lim_{T\to\infty} T^{2\alpha} E\left|f_T^*(w) - f(w)\right|^2 = \frac{1}{B}\int k^2(z)dz\left|f(w)\right|^2\left\{1 + \delta(0, w)\right\} + B^{2r}\left|k^{(r)}f^{(r)}(w)\right|^2. \tag{6.5}$$

7. *Strong Consistency and Asymptotic Normality.*—It may be shown that the estimate $f_T^*(w)$ is strongly consistent (that is, consistent with probability 1) if its order of consistency is $T^{2\alpha}$, where $1/4 < \alpha < 1/2$. Under some additional assumptions, it may also be shown that $f_T^*(w)$ is asymptotically normal.

* This work was supported by the Office of Naval Research under Contract N6-ONR-27135 Reproduction in whole or in part is permitted for any purpose of the United States Government.

[1] M. S. Bartlett, "Periodogram Analysis and Continuous Spectra," *Biometrika*, **37, 1950**; U. Grenander, "On Empirical Spectral Analysis of Stochastic Processes," *Arkiv Mat.*, **1, 1951**; M. S. Bartlett, *An Introduction to Stochastic Processes* (Cambridge, England: At the University Press, 1955), pp. 274–288.

ON WEYL'S IMBEDDING PROBLEM

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1. On a two-dimensional manifold S which is of the topological type of a sphere, let there be given a positive definite Riemannian metric $ds^2 = g_{ik}(u^1, u^2)\, du^i\, du^k$, where the parameters u^1, u^2 are local co-ordinates on S, valid in some neighborhood of any given point P of S. If B is any property of smoothness (such as analyticity), then by $(g_{ik})\ \varepsilon\ B$ will be meant that, in a neighborhood of any P, the components of

the metric tensor satisfy the condition $g_{ik}(u, v) \in B$ when the parameters $u = u^1$, $v = u^2$ are suitably chosen on that neighborhood. It will always be assumed that the given metric is such as to possess a ("Gaussian") curvature $K = K(u, v)$ (in an appropriate sense) and that K is continuous and positive on S. In this and only in this case will the matrix notation (g_{ik}) be used for a metric on S.

Let a metric (g_{ik}), belonging to a ds^2 given on S, be called a *W-metric* if there exists in the Euclidean (x, y, z)-space a surface T of class C^2 which is topologically equivalent to S and on which the first fundamental form becomes the given ds^2. If this is symbolized by $(g_{ik}) \in W$, then Weyl's imbedding problem consists in the determination of smoothness properties of (g_{ik}) which are sufficient for $(g_{ik}) \in W$.

It will be explained in Sections 2 and 3 why, in the definition of a *W-metric*, the smoothness property imposed on imbedding T is precisely $T \in C^2$. Section 4 will contain the formulation of the result of this paper, a result supplying a condition which, when satisfied by a (g_{ik}), assures that $(g_{ik}) \in W$. The proof of the theorem will be given in Sections 5 and 6.

2. Let $(g_{ik}) \in R$ mean that the metric (g_{ik}) on S is what today is usually referred to as a regular metric.[1] In other words, $(g_{ik}) \in R$ means that $(g_{ik}) \in C^1$ and that, in addition, (g_{ik}) possesses a continuous curvature K. Thus $(g_{ik}) \in C^2$ is sufficient for $(g_{ik}) \in R$. But this sufficient condition fails to be necessary for $(g_{ik}) \in R$. In fact, if (g_{ik}) is the matrix of the first fundamental form on any $T \in C^2$, then $(g_{ik}) \in R$ is always true (the K of (g_{ik}) being the Gaussian curvature of the surface T), but $(g_{ik}) \in C^2$ need not hold.

In his classical paper Weyl[2] claimed, though he did not claim to have proved, that the condition $(g_{ik}) \in R$ is (not only necessary but also sufficient) for the existence of a $T \in C^2$ on which (g_{ik}) is the matrix of the first fundamental form. In the nomenclature of Section 1, this existence statement can be expressed as follows: $(g_{ik}) \in R$ implies that $(g_{ik}) \in W$ (or, what is the same thing, that $W = R$). But it was shown recently that this conjecture of Weyl is false.[3]

A condition on (g_{ik}) which is sufficient for $(g_{ik}) \in W$ was first proved by H. Lewy.[4] In fact, he proved that, if A refers to the class of analyticity, then $(g_{ik}) \in A$ is sufficient in order that there exist for (g_{ik}) an *analytic* imbedding T. Since this implies that $T \in C^2$, it follows, as a corollary, that $(g_{ik}) \in A$ is sufficient for $(g_{ik}) \in W$.

3. This corollary of Lewy's theorem suffices to conclude the full force of his theorem. In fact, $(g_{ik}) \in A$ implies that $K \in A$ holds for the curvature of (g_{ik}). But according to a theorem on elliptic partial differential equations which, for the particular case at hand (Monge-Ampère), was first proved by Pogoreloff,[5] it follows from $T \in C^2$ and $K \in A$ that T is an analytic surface.

Similar conclusions hold if the A-class is replaced by appropriate classes of differentiability. In order to specify these conclusions (which, along with the preceding A-conclusion, justify the assignment of the particular C^2-class of imbeddings T in the definitions of $(g_{ik}) \in W$, given in Sec. 1), the following notation will be used: Let $r \geqq 0$, and put $n = [r]$. If $r \neq n$, then the class C^r will refer to that subclass of C^n in which the nth derivatives satisfy a uniform Hölder condition of index $r - n$. Thus, if $r \geqq 2$ (hence $r \geqq n \geqq 2$), then $(g_{ik}) \in C^r$ implies that (g_{ik}) possesses a $K \in C^{r-2}$.

Conversely, suppose that there is given a surface $T \in C^2$ on which the Gaussian curvature K is positive and, in terms of local parameters (u, v) which supply a

C^2-parametrization of the surface, $K \in C^{r-2}$, where $r \geqq n \geqq 2$ and $K > 0$. Then it is known[6] that $T \in C^r$ provided that $r \neq n$. Since the Gaussian curvature of a $T \in C^2$ is the same as the K of the first fundamental form (which is a regular metric), it follows that if the K of $(g_{ik}) \in W$ satisfies the condition $K \in C^r$, where $r > n \geqq 2$, then the assertion $T \in C^2$ (concerning that surface T the existence of which is equivalent to the assumption $(g_{ik}) \in W$) improves itself to the assertion $T \in C^r$.

4. Nirenberg[7] succeeded in extending Lewy's result, quoted in Section 2, so as to dispose of the analyticity of (g_{ik}), as follows: If $r > n \geqq 4$ and $(g_{ik}) \in C^r$, then (g_{ik}) possesses an imbedding $T \in C^r$. In view of the end of Section 3 and of the definition of the W-class (Sec. 1), this result is equivalent to the following statement: $(g_{ik}) \in C^r$ is sufficient for $(g_{ik}) \in W$ if $r > 4$.

The purpose of this note is to improve on this result of Nirenberg, by omitting the Hölder restriction which the inequality $r > 4$ imposes on the fourth derivatives of the metric tensor—in other words, to prove the following theorem on the existence of an imbedding $T \in C^2$ (which, in view of the end of Sec. 3, must necessarily satisfy the condition $T \in C^3$, and even the condition $T \in C^t$ for every $t < 4$):

THEOREM. *If* $(g_{ik}) \in C^4$, *then* $(g_{ik}) \in W$.

It remains undecided whether the C^4 in this theorem can be improved to C^3 or, for that matter, to C^2 (all that is sure is that it cannot be improved to R, a condition somewhat more inclusive than C^2; cf. the counterexample referred to in Sec. 2 above). The reason why the method of proving the implication $(C^r) \to (W)$ fails to apply to any $r < 4$ is that Weyl's inequality[8] involves the second derivatives of the Gaussian curvature.

5. The assumption $(g_{ik}) \in C^4$ implies that there exists on S a sequence of metrics (g_{ik}^1), (g_{ik}^2), . . . having the following pair of properties: (i) if A refers to the class of analytic metrics, then $(g_{ik}^j) \in A$ holds for $j = 1, 2, \ldots$ in terms of the local parameters (u, v) of the three given functions $g_{ik} = g_{ik}(u, v)$ on S; and (ii) if $0 \leqq h \leqq 4$ and $j \to \infty$, then $D^h(g_{ik}^j) \to D^h(g_{ik})$ holds uniformly on S (here D^h denotes any of the $h + 1$ differentiation symbols $\partial^{\alpha+\beta}/\partial u^\alpha \partial v^\beta$, where $\alpha + \beta = h$).

The existence of a sequence of metrics satisfying properties (i) and (ii) is nothing but the approximation theorem of Weierstrass-Tonelli (for $h = 0$, of Weierstrass himself) for the sphere S. Correspondingly, the existence of the approximating sequence (g_{ik}^1), (g_{ik}^2), . . . follows with the aid of appropriate "singular integrals" (Lebesgue) on S.

It is the use of such approximations that will allow a modification of Nirenberg's proof in such a way that his assumption $r > 4$ in the implication $(C^r) \to (W)$ becomes improved to $r = 4$. At the same time, there results a proof (for the non-analytic case) which is based on Lewy's implication, that is, on $(A) \to (W)$. Incidentally, it will be clear from the proof that, instead of Lewy's implication, that of Nirenberg could also be used (simply by replacing $(g_{ik}^j) \in A$ in property (i) by $(g_{ik}^j) \in C^r$, where $r > 4$).

6. First, it follows from property (i) and from Lewy's theorem that there belongs to every j a closed analytic surface, say T^j, on which (g_{ik}^j) is the matrix of the first fundamental form. Let K^j denote the Gaussian curvature on T^j. Then it follows from property (ii) that $K^j \to K$ holds uniformly on S, as $j \to \infty$. Since K and K^j are positive and continuous on S, this implies that $c < K^j < 1/c$ holds for a certain $c > 0$ which is independent of j and of the position on S.

Let $X = X^j(u, v)$, where $X = (x, y, z)$, be the parametrization of the surface T^j (in terms of the local parameters (u, v) to which the metric tensors are referred). Then $X^j = X^j(u, v)$ is a vector function of class C^4 (in fact, it is analytic). It follows, therefore, from assumption (ii), from the existence of uniform bounds for K^j and $1/K^j$, and from Weyl's inequality, that the vector functions $X^1, X^2, \ldots$ and their first and second partial derivatives are uniformly bounded on S.

The crucial point is that the second derivatives of these vector functions are equicontinuous on S. In fact, the three second partial derivatives $D^2X^j(u, v)$ satisfy a uniform Hölder condition on S, with a Hölder factor and a Hölder exponent which are independent of j. This is the fundamental observation of Nirenberg derived by him from the general theory of partial differential equations of elliptic type.[9]

Since the functions $X^j(u, v)$ and their first *and* second partial derivatives are uniformly bounded and equicontinuous on S, the sequence $X^1, X^2, \ldots$ contains a subsequence having the following properties: If the jth element of that subsequence is denoted simply by X^j, then there exists on S a vector function $X = X(u, v)$ of class C^2 satisfying $X^j \to X$, $DX^j \to DX$, and $D^2X^j \to D^2X$, as $j \to \infty$, uniformly on S. In view of assumption (ii) (an assumption which is now needed only for $h \leqq 2$ rather than for $h \leqq 4$), this implies that T: $X = X(u, v)$ defines a closed surface $T \in C^2$ on which (g_{ik}) is the matrix of the first fundamental form. This proves the theorem italicized in Section 4.

[1] For references cf. A. Wintner, *Am. J. Math.*, **77**, 849, 1955.
[2] H. Weyl, *Vierteljahrschr. naturforsch. Ges. Zürich*, **61**, 40–72, 1915.
[3] A. Wintner, *Am. J. Math.*, **78**, 117–136, 1956.
[4] H. Lewy, these PROCEEDINGS, **24**, 104–106, 1938.
[5] A. B. Pogoreloff, *Mat. Sbornik*, **73**, 88–103, 1952.
[6] P. Hartman and A. Wintner, *Am. J. Math.*, **75**, 612–613, 1953.
[7] L. Nirenberg, *Communs. Pure and Appl. Math.*, **6**, 337–394, 1953.
[8] Weyl derived his identity and the resulting inequality at the end of his paper (*op. cit.*). A simple proof was given by S.-S. Chern, *Duke Math. J.*, **12**, 279–290, 1945. For a detailed discussion of Weyl's identity and inequality cf. A. Wintner, *Ann. di Mat.*, 1956 (to appear).
[9] L. Nirenberg, *op. cit.*, pp. 374–375 and 103–156.

THE BEHAVIOR IN TRANSPLANT OF LYMPHOCYTIC NEOPLASMS ARISING FROM PARENTAL THYMIC GRAFTS IN IRRADIATED, THYMECTOMIZED HYBRID MICE

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INTRODUCTION

Lymphocytic neoplasms arise in the thymus in certain strains of mice, either spontaneously or as a result of treatment with X-radiation, carcinogenic hydrocarbons, and estrogens. In several instances, notably the spontaneous occurrence in AK[1] and AKR,[2] in methylcholanthrene induction in DBA,[3] and in the radiation-induced disease in C57BL mice,[4] total thymectomy strikingly reduces the incidence of the disease.

In certain instances the presence of a thymic graft in the thymectomized animal has partially restored the host potentiality of developing lymphocytic neoplasms. In strain DBA/2, thymectomy, followed by autologous thymic grafting and methylcholanthrene painting, respectively, has restored the host potentiality for the development of lymphocytic neoplasms.[5]

In another instance, under different conditions and employing strain C57BL mice, Kaplan[6] reported that thymectomy, followed by a course of X-radiation and the implantation of a nonirradiated thymus, restored this potentiality. It was further specified that lymphoid tumors arose in thymectomized mice which had received a nonirradiated thymic graft within 2 hours to 8 days following the last exposure of the course of fractionated X-radiation. Further work by Kaplan and Brown[7] reported that the origin of these lymphoid tumors, in certain of the animals, was the thymic graft itself.

These results were interpreted to mean that lymphocytic neoplasms arose in the thymic grafts which received no irradiation at any time, indicating that the initiation of leukemogenesis did not require actual exposure to X-radiation but rather required an exposure to a postradiation influence.

In our experiments an attempt was made to study, in more detail, the relationship of the grafted thymus to this postradiation influence by employing a genetically labeled thymic graft. The thymic graft was obtained from one of the parental strains of a hybrid mouse. In this case the parental thymus can successfully grow in the compatible but genetically different host.[8] The Kaplan subline of the C57BL strain was crossed with A/He strain mice; each of these strains carry different H-2 alleles,[9] thus assuring that the F_1 hybrid mouse would have strong tissue antigens that neither parent strain possessed. Although progressively growing neoplasms from either parental strain would be expected to kill (C57BL × A)F_1 hybrid mice, such neoplasms arising in tissues of these specific F_1 mice would not be expected to grow progressively in either A or C57BL mice.

As a control study, neoplasms originating in the C57BL/Ka and A/He strains, and in (C57BL/Ka × A)F_1 mice, were transplanted into the two parental strains and the F_1 hybrids to establish the incompatibility of the F_1 tumors in the parental strains and the compatibility of parental neoplasms in the F_1 hybrid. These neoplasms included two pulmonary tumors of A/He strain origin (11416 at the fifty-third transfer generation and C4461 at the one hundred and seventy-first transfer generation), four X-ray-induced lymphocytic neoplasms which arose in intact C57BL/Ka mice (L7057, L7058, L7098, and L7134 at the fifteenth to twenty-third transfer generations), and seven of the X-ray-induced lymphocytic neoplasms originating in (C57BL × A)F_1 hybrid mice listed in Table 2. The results of these experiments are given in Table 1 and show precisely what is expected according to

TABLE 1

Type of Neoplasm	Progressive Growth Resulting in Death (+) in			
	No.	C57BL	A	(C57BL×A)F_1
C57BL lymphocytic neoplasms	4	4(+)	?	4(+)
A pulmonary tumors	2	Neg.	2(+)	2(+)
(C57BL×A)F_1 lymphocytic neoplasms	7	Neg.	Neg.	7(+)

the genetic laws of transplantation.[10] Parental tumors grew progressively in the strain of origin and in hybrids involving the AKR strain, whereas the F_1 lympho-

cytic neoplasms grew progressively only in the genotype carrying the essential isoantigens of both parental types, the F_1 hybrid.

MATERIALS AND METHODS

C57BL/Ka females of the F 13–15 inbred generations (in this laboratory) were mated with A/He[11] males of the F 90–96 inbred generations. The pedigreed F_1 hybrids were weaned at approximately 30 days of age, and 5–8 animals, as complete litters, were placed in cages and fed Derwood dog chow ad libitum. Three groups of mice were then designated, and the lymphocytic neoplasms arising in the various groups were the source of the transplantation material studied. Group I remained intact and received X-ray only, as described below. Group II was thymectomized and then irradiated. Group III was thymectomized, irradiated, and then grafted. A fourth group of 36 intact, nonirradiated, nongrafted mice have been observed for 10–11 months. Thirty-four are living and well; no lymphocytic neoplasms have been observed. The F_1 mice were thymectomized between 30 and 47 days of age by a method, previously described,[1, 2] employing intraperitoneally administered nembutal anaesthesia (80 mg/kg of body weight). The thymus was delivered into the sternal wound by the aid of a glass suction tip; suction was created by negative pressure from the laboratory vacuum outlet. In this way both lobes could be easily grasped and with gentle traction removed.

Within 1–7 days following thymectomy, the age of the mice then being 30–54 days, a course of four weekly exposures to total-body X-radiation was begun; each exposure was to 225 r, and the total dose was 900 r.[12] Following the last exposure, Group III was subdivided into two groups. One group received a thymic implant on the day of the last exposure (first day); the second group received a thymic implant 7 days following the last exposure.

Thymuses obtained from 1–12-day-old C57BL/Ka pedigreed litters were implanted subcutaneously in the right flank near the axilla by means of a 12- or 13-gauge trocar. A few thymic grafts were obtained from (C57BL × A)F_1 pedigreed litters. Each recipient received a whole thymus, and each thymic graft was recorded as to the source and recipient. This procedure was carried out with the usual aseptic precautions; however, bacteria were inadvertently introduced. In order to obviate contamination of the grafts, a course of penicillin and streptomycin was given to the recipients; 1–4,000 units of penicillin/mouse/day and 2.5–5 mg streptomycin/mouse/day were employed. No standard schedule was used throughout; treatment was usually continued until 10–14 days after grafting.

The animals were examined weekly for evidence of lymphocytic neoplasms. These neoplasms in the intact mice appeared as progressive enlargement of the spleen and lymph nodes, with thymic enlargement and respiratory embarrassment.

Diagnosis of lymphocytic neoplasms in the thymectomized, grafted mice was made by progressive enlargement of the graft. When evidence of lymphocytic neoplasm presented itself, the animal was sacrificed by cervical dislocation, and transplantation was accomplished. In cases of graft involvement, enlargement was usually progressive over a period of 7–14 days. All grafts attained a size of 7–10 mm. in diameter. In some cases, however, grafts which became neoplastic later in the course of the experiment attained a large size (approximately 1 × 1 cm.) gradually, and dissemination was not observed over a period of 2–3 weeks.

Each animal was autopsied, and representative tissues (6–10 sections) were fixed in Tellyesniczky's fluid (70 per cent ethyl alcohol, 20 parts; formalin, 2 parts; glacial acetic, 1 part) and stained with hematoxylin and eosin. The pathologic diagnoses of lymphocytic neoplasms were confirmed by Dr. Thelma Dunn and classified according to the classification made by Dunn.[13]

The method of transplanting the neoplastic grafts was by use of a trocar, as routinely practiced for grafting solid tumors in this laboratory. The thymic graft, which had become neoplastic, was used as the sole tissue source of the transplant.

Following the initial transfer, neoplastic grafts of F_1 origin were subcutaneously transplanted to a second or a third generation. In each case their behavior was identical with the original pattern.

RESULTS

Table 2 reveals the number of animals employed in the experiment and the time of origin of the various lymphocytic neoplasms. Twenty of the 32 intact, irradiated

TABLE 2

LYMPHOCYTIC NEOPLASMS IN (C57BL×A)F_1 MICE FOLLOWING X-RADIATION

GROUP	TOTAL NO.	NO. DYING OF LYMPHOCYTIC NEOPLASM (Month)									TOTAL NO. DEAD WITH LYMPH. NEOP.	TOTAL NO. DEAD WITHOUT LYMPH. NEOP.	NO. LIVING*
		4	5	6	7	8	9	10	11	12			
I, intact	32	0	2	3	5	6	1	2	1	0	20	0	12
II, thymectomized	47	0	0	0	0	0	0	0	0	0	0	5	42
III, thymectomized:													
Plus 1st-day graft	38	0	2	0	0	4	1	1†	..	..	8†	5	25
Plus 7th-day graft	37	0	2	0	1	1	2	..	..	..	6	3	28
Total	75	0	4	0	1	5	3	1			14	8	53

* Average age of mice alive is 12 months.
† This animal developed a generalized lymphocytic neoplasm not involving the graft.

mice have developed lymphocytic neoplasms. Seven of these neoplasms were transplanted and grew only in F_1 hybrids (see Table 1). Total thymectomy has completely prevented the appearance of lymphocytic neoplasms. Gross appearance of the twenty induced neoplasms in the intact, irradiated controls revealed that the thymus was markedly involved in 18 of these mice. The neoplasms listed in the thymectomized, grafted group represent neoplasms which arose in thymic grafts. In these cases, the first gross evidence of lymphocytic neoplasm appeared as progressive enlargement of the graft. It is of interest to note that under the conditions of this experiment there was little difference between the incidence of neoplasms arising from thymic grafts in the first-day and in the seventh-day groups.

Tables 3 and 4 describe the history of the individual animals in which thymic grafts gave rise to lymphocytic neoplasms and the details of the behavior of these neoplasms in transplant.

Lymphocytic neoplasms arising in C57BL thymic grafts at 5 and $5^1/_2$ months, in both first-day and seventh-day postirradiation groups, grew progressively in C57BL and (C57BL × A)F_1 hosts. These neoplasms behaved in transplant as C57BL in origin (see also Table 1). In contrast, lymphocytic neoplasms arising in C57BL thymic grafts later, at 7–10 months, grew progressively following transplantation only in (C57BL × A)F_1 hybrids. These neoplasms indicated their

TABLE 3

HISTORY OF ANIMALS BEARING NEOPLASTIC THYMIC GRAFTS

Tumor No.	Sex	Age Thymectomized (Days)	Age Began X-ray* (Days)	Age Grafted (Days)	Graft No.†	Donor Strain of Thymus	Age Implanted (Days)	Age Lymphocytic Neoplasm Developed (Months)	Local (L) or Generalized (G) Neoplasm	Transplant Character‡
1st day:										
286	♂	39	42	62	IV	C57BL	1	5½	G	C57BL
291	♀	38	40	62	IX	C57BL	1	5	G	C57BL
351	♀	36	38	60	I	C57BL	4	8	L	(C57BL×A)F₁
3521	♀	39	42	62	IV	C57BL	1	8	G	(C57BL×A)F₁
3904	♀	32	34	56	VII	C57BL	2	8	L	(C57BL×A)F₁
3907	♀	32	34	56	VII	C57BL	2	8	G	(C57BL×A)F₁
468	♀	38	39	61	VIII	C57BL	1	9½	L	(C57BL×A)F₁
7th day:										
348	♀	31	37	65	XIV	C57BL	1–5	5½	G	C57BL
349	♀	31	37	65	XIII	C57BL	1	5½	L	C57BL
3906	♀	32	38	66	XII	C57BL	1	7	G	(C57BL×A)F₁
3905	♂	47	54	82	XI	(C57BL×A)F₁	3	9	L	(C57BL×A)F₁
467	♀	32	38	66	XIII	C57BL	1	8½	G	(C57BL×A)F₁
5345	♀	31	31	59	XIV	C57BL	1–5	10	G	.....

* Thymectomy done 1–7 days prior to first exposure to X-ray.
† Roman numeral refers to litter source; thus all IV grafts were brother and sister donors from one litter.
‡ Histocompatibility characteristics: C57BL = grew progressively in the C57BL and F₁ mice; (C57BL×A)F₁ = grew progressively only in F₁ mice (see Table 4).
§ This neoplasm was classified as Type B by Dr. Dunn.

origin from F_1 cells, although the site of origin was from a C57BL thymic graft. It is of interest to note that two pairs of lymphocytic neoplasms, Nos. 286 and 3521 and Nos. 349 and 467 (Table 3), presented different histocompatibility characteristics. Nevertheless, the donors in these cases were C57BL litter mates. Thus, employing as closely related thymic tissue as possible, the two different histocompatibility types of neoplasms were found.

DISCUSSION

Total thymectomy in the irradiated (C57BL × A)F_1 hybrid mouse drastically reduced the incidence of lymphocytic neoplasms. Transfer of C57BL thymic tissue to the subcutaneous connective tissues partially restored the potentiality of developing lymphocytic neoplasms. It is especially clear that the site of origin of the lymphocytic neoplasms was the thymic graft. The first evidence of the neoplasm in X-rayed, thymectomized mice receiving a thymic graft was found by palpation of the graft, which enlarged progressively during the last 2 weeks of observation. Furthermore, in five cases the neoplasms were confined to the thymic graft; no evidence was found of infiltration to other organs.

It has been postulated by Kaplan and Brown that the progenitor cells of the thymic-graft neoplasms had not been

exposed to X-radiation, thus establishing "the existence of a completely indirect mechanism of induction of lymphoid tumors in systemically irradiated mice."[7] Under the experimental conditions employed by these investigators, this postulate could not be established, since there was no way of differentiating between nonexposed cells of the thymic graft and exposed cells of the host which might populate the graft. C57BL thymic tissue was grafted into C57BL mice. It is possible, however, to differentiate between cells exposed to X-radiation and those not exposed by employing a tissue which is compatible, that is, which will grow in the host, yet is genetically different from the host. In this study a compatible C57BL thymus was grafted into the compatible, but genetically different, (C57BL × A)F_1 hybrid mouse.

A similar type of experiment has been reported from this laboratory, employing AKR thymic fragments grafted into (C3H × AKR)F_1 mice.[5] In this situation the compatible graft (AKR), in the compatible, but genetically different, host (AKR × C3H), underwent a change in the thymic lymphocyte population, which was observed histologically and later confirmed by transplantation studies. These fragments became neoplastic and, upon transfer, did not grow progressively in AKR mice, the strain of origin

TABLE 4

BEHAVIOR IN TRANSPLANT OF THE INDIVIDUAL NEOPLASMS OF THYMIC GRAFT ORIGIN
Progressive Growth (+) Resulting in Death; No Growth (−)

Tumor No.	Source of Graft	Transfer	C57BL			Average Day at Death	(C57BL×A)F_1		Average Day at Death	Final Histocompatibility Type
			No.	No. +	No. −		No.	No. +	No. −	
286	C57BL	1	3	3	0	37.5	3	3	0	90.0	C57BL
291	C57BL	1	3	3	0	30	3	3	0	60	C57BL
348	C57BL	1	3	3	0	32.7	3	3	0	51.6	C57BL
349	C57BL	1	3	3	0	57	3	3	0	63	C57BL
351	C57BL	1	3	0	3	. . .	3	2*	1	62	(C57BL×A)F_1
		3	3	0	3	. . .	3	3*	0	36	(C57BL×A)F_1
		1	3	0	3	. . .	3	3*	0	48	(C57BL×A)F_1
		2	3	0	3	. . .	2	2	0	57	(C57BL×A)F_1
3521	C57BL	1	4	0	4	. . .	4	4*	0	91	(C57BL×A)F_1
		2	3	0	3	. . .	2	2*	0	. . .	(C57BL×A)F_1
3904	C57BL	1	3	0	3	. . .	2	2*	0	77	(C57BL×A)F_1
		2	4	0	4	. . .	4	2*	0	43	(C57BL×A)F_1
3907	C57BL	1	3	0	3	. . .	2	2	0	55	(C57BL×A)F_1
		2	3	0	3	. . .	3	3*	0	46	(C57BL×A)F_1
3906	C57BL	1	3	0	3	. . .	2	2	0	108	(C57BL×A)F_1
		2	3	0	3	. . .	3	1	2	. . .	(C57BL×A)F_1
3905	(C57BL×A)F_1	1	3	0	3	. . .	3	1	2	48	(C57BL×A)F_1
468	C57BL	1	3	0	3	. . .	3	3	0	. . .	(C57BL×A)F_1
		2	3	0	3	. . .	3	3	0	. . .	(C57BL×A)F_1
467	C57BL	1	3	0	3	. . .	3	3	0	. . .	(C57BL×A)F_1

* One mouse was sacrificed for transplant and is not included in calculation of average.

of the thymic tissue, but grew only in the specific F_1 recipient mice used.

In the present experiment, employing the techniques of Kaplan and Brown but using a marked thymic graft (C57BL) in a compatible, but genetically different, host, it was possible to show that the postulate of Kaplan and Brown was, in part, correct. That is, cells of C57BL origin never exposed to X-radiation did give rise to lymphocytic neoplasms, but only in four of twelve cases. Thus this situation represented only part of the complete picture. A greater number of grafts which developed into lymphocytic neoplasms developed from cells derived from the irradiated host animal.

It cannot be established from this experiment whether the cells which were to give rise later to the neoplastic population (the progenitor cells), in the case of the (C57BL × A)F_1 lymphocytic neoplasms, were the actual cells exposed to the radiation or whether they were division products of such cells. It should be noted that, regardless of whether the cell or cells were actually exposed or were division products of such cells, grafted thymus, transplanted as late as 7 days after irradiation, is capable of developing these cells toward a neoplastic state. The precise role of grafted thymus is not further elucidated as yet. It can be said that the thymus must be present; yet, paradoxically, in many cases it does not supply the progenitor cell.

It is possible that the thymus behaves in this experimental situation as do certain embryonic tissues, that is, by direct tissue interaction. Thus the grafted thymus represents a "sphere of influence." A similar type of action has been demonstrated in other experiments from this laboratory, employing the (C3H × AKR)F_1 hybrid mouse. In these experiments it was found that AKR thymus was a source of lymphocytic neoplasms in this hybrid, while the equally compatible C3H thymus was not. The "sphere of influence" appears to be genetically determined and is specific to certain strains. Still further, employing AKR grafted spleen in these hybrids, the effect was again not equal to the thymus.[14]

Tissue interdependence in carcinogenesis has been observed recently[15] in methylcholanthrene-induced skin tumors in mice. Here a relationship to the induction of tumors has been established between epidermis and the deeper-lying tissues; however, the specific mechanisms would appear not to be of the same nature as those involved in our experiments. Nonetheless, the general principle of tissue interdependence may well be of great importance in the evolution of neoplastic cells.

The present experimental model provides an opportunity for the investigator to study with great accuracy the events involved in the evolution of the leukemic cell. The specific role of the thymus in leukemogenesis, the age of thymic graft in relation to age of irradiated host, the mechanism of tissue interaction or interdependence in the induction of lymphocytic neoplasms (and possibly other morphologic forms of lymphomas), and characterization of the postirradiation influence are some of the problems which lend themselves to investigation by the use of such a model.

CONCLUSIONS

1. Thymectomy prevents the development of lymphocytic neoplasms in (C57BL × A)F_1 hybrid mice which have received a course of fractionated X-radiation.

2. Twenty of 32 irradiated, nonthymectomized (C57BL × A)F_1 hybrids developed lymphocytic neoplasms.

3. The presence of thymic grafts obtained from 1–12-day-old C57BL donors partially restores the ability of the irradiated host to develop lymphocytic neoplasms.

4. The site of origin of the lymphocytic neoplasms in thymectomized, irradiated, thymic-grafted mice is the grafted thymus.

5. The progenitor cells of the lymphocytic neoplasms arising in the thymic grafts studied in this experiment were contributed from either the (C57BL × A)F_1 host or, in a smaller number of cases, from the nonirradiated C57BL graft, as determined by the histocompatibility character of the resulting neoplasm.

6. A discussion is given of this experimental approach to the time and site of origin of the leukemic cell and the mechanisms involved in leukemogenesis.

* National Institutes of Health, Public Health Service, Department of Health, Education, and Welfare.

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[8] G. D. Snell, "Transplantable Tumors," chap. xiv of *The Physiopathology of Cancer*, ed. F. Homburger and W. H. Fishman (New York: Paul B. Hoeber, Inc., 1953).

[9] G. D. Snell, P. Smith, and F. Gabrielson, *J. Nat. Cancer Inst.*, **14**, 457, 1953; G. Hoecker, S. Counce, and P. Smith, **40**, 1040, 1954; G. Hoecker, personal communication, 1955.

[10] Recent studies in the field of histocompatibility genetics have revealed certain exceptions which are of rare occurrence. Adequate interpretations have been given for such exceptions (see T. S. Hauschka, *Cancer Research*, **12**, 615, 1952). They in no way detract from the generally accepted laws of tissue transplantation.

[11] The pulmonary tumors (see Table 1) and the A/He males used in establishing hybrid crosses were kindly supplied by Dr. W. E. Heston of the National Cancer Institute. C57BL/Ka mice were obtained originally from Dr. Henry Kaplan, of Stanford University, and have been bred brother × sister in this laboratory for the number of generations indicated.

[12] Physical factors of irradiation were two (opposing) tubes above and below the mice, each 54 cm. from the mouse target; 200 kvp., 14 ma. (each tube); beam filtration, 0.25 cm. Cu and 0.55 cm. Al; 225 r was delivered in 1.75 minutes.

[13] T. B. Dunn, *J. Nat. Cancer Inst.*, **14**, 1281, 1954.

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*PISCIVOROUS GASTROPODS OF THE GENUS CONUS**

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Many marine snails are known to feed saprophytically on dead fishes.[1, 2] The observations reported here are believed to represent the first known instances of the attacking, killing, and eating of live fishes by gastropod mollusks. It will also be shown that live fishes are the dominant and apparently exclusive food of one, and probably both, species of *Conus* to be discussed. This paper thus reports on an ecological niche not previously known to be occupied by gastropods.

Feeding Mechanism.—Members of the family Conidae (Prosobranchiata: Stenoglossa) are predacious and feed by injecting a venom into the prey organism with a detachable, dartlike radula tooth (Fig. 1). There are but few reports of the observation of this process in the living animal. Most of our knowledge has been gained from anatomical studies of fixed specimens[3] and from accounts of human injuries inflicted by these snails.[4] A more complete account of the feeding process, based on current studies of the ecology of the group, will be published elsewhere.

Food in Nature.—Fish remains were first discovered in the alimentary tract of a specimen of *Conus striatus* L. from Hawaii dissected by Kondo.[4] Additional fish remains were found in several specimens of this species from Hawaii and Guam dissected by the present author.[5] In each case, however, remains were insufficient to permit identification of the fishes, even to family. No other types of prey organism were found in alimentary tracts of *C. striatus*.

No previous reports are known of fish remains in *C. catus* Bruguière. Examination of alimentary tracts of twenty specimens of this species collected by the author in Hawaii revealed bones, otoliths, scales, and other remains of fishes in nine. The alimentary tract of one contained a discharged radula tooth, and those of the remaining ten specimens were empty.

Three of the fishes were identified as the blenny *Istiblennius gibbifrons* Quoy and Gaimard. Two others are probably *Bathygobius fuscus* Rüppell. The remains of four were insufficient to permit identification.

Feeding of C. striatus in the Laboratory.—Specimens of *C. striatus* have been fed *Kuhlia sandvicensis* Steindachner, *I. gibbifrons* Quoy and Gaimard, and *Entomacrodus marmoratus* Bennett in the laboratory.

Abbreviations: *b* = barbs; *k* = terminal knob; *l* = ligament; *m* = mouth; *p* = proboscis; *rt* = position of radula tooth within proboscis; *s* = siphon; *sh* = shell; *t* = tentacle, which bears the eye.

Fig. 1.—Photograph of radula tooth of *Conus striatus*. Length of tooth, excluding ligament, 10.6 mm.

Fig. 2.—Drawing of *C. striatus* with proboscis extended. External organs labeled.

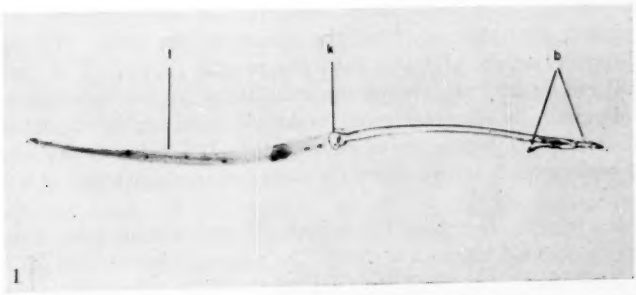
Fig. 3.—Feeding process in *C. striatus*. Proboscis is extended toward fish (*Entomacrodus marmoratus*).

Fig. 4.—Feeding process in *C. striatus*. Proboscis in contact with fish.

Fig. 5.—Feeding process in *C. striatus*. Fish is stung and impaled on ejected radula tooth.

Fig. 6.—Feeding process in *C. striatus*. Fish is drawn into distended buccal cavity.

Fig. 7.—Feeding process in *C. striatus*. Engulfment is complete.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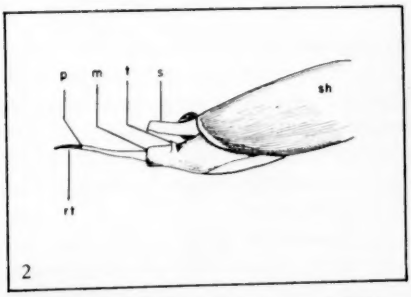
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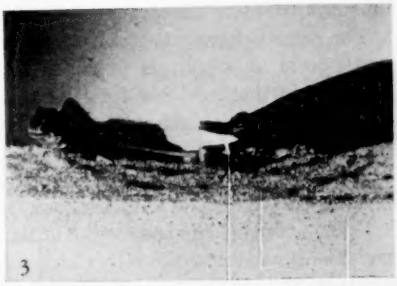
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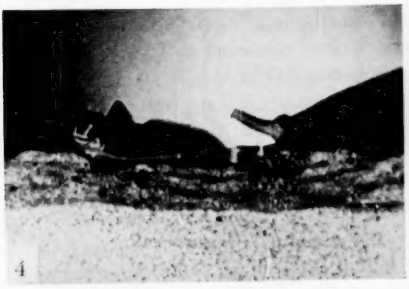
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Conus striatus normally occurs on sandy bottoms, on shallow reef platforms as well as in deeper water, throughout the Indo-Pacific region. A specimen placed in an aquarium will bury itself almost completely if sufficient sand is provided. Only the siphon may protrude from the surface of the sand. The siphon continually directs a stream of water into the mantle cavity. This water passes over the gill and osphradium, and by the anus, before leaving the mantle cavity.

The osphradium, a chemoreceptory organ, is more highly specialized in the Conidae than in any other family of gastropods.[1] It is believed that chemoreception is the sensory modality by which the snail first becomes aware of a fish placed in its tank.

Less than a minute after the fish is introduced, and in some cases within several seconds, the gastropod becomes active. The proboscis is extended up out of the sand, and movements of the foot begin raising the entire animal, although the eyes are initially below the sand-water interface.

The same behavior pattern is noted following introduction, not of a fish, but of water from an aquarium in which fish have been swimming. This observation provides further evidence for the hypothesis that the initial stimulus is chemical rather than visual. That mechanical stimulation is not the initial releaser of proboscis activity is indicated by the absence of response following agitation of the water with a glass rod. Recently killed fish (*K. sandvicensis*) did not evoke this behavior pattern, even when left in the tank for 24 hours.

When a live pelagic fish (e.g., *Kuhlia*) is introduced into the tank with the gastropod, the proboscis of the latter is directed vertically and, in a specimen measuring 115 mm. in shell length, may be extended as much as 60 mm. The single radula tooth is held within the lumen of the proboscis, its point slightly posteriad of the aperture (Fig. 2). The tip of the proboscis is often observed to "track" or follow the fish when the latter swims past within a centimeter or so of it. However, the stimulus for release of the dart is tactile, the dart not being ejected until the body of the fish comes in contact with the tip of the proboscis (Fig. 4).

When injected into the fish, the dart is not completely freed from the proboscis, but its posterior end is held tenaciously within the latter, apparently by circular muscles which contract anteriad of the terminal knob (Fig. 1) at the posterior end of the dart. The radula tooth, then, serves as a harpoon, and the impaled fish is rapidly drawn toward the mouth by longitudinal contraction of the proboscis.

Simultaneously the venom, apparently a powerful neurotoxin, quickly paralyzes the fish, which ceases struggling within several seconds. Also simultaneously, the diameter of the mouth expands from a few millimeters to more than two centimeters in order to receive the meal (Fig. 6). After engulfment (Fig. 7), the buccal cavity remains greatly distended for several hours, as the prey must be partially digested before being passed on to the less extensible regions of the alimentary tract.

In dissecting specimens of *C. catus* which had recently fed, it was noted that fishes in the pharynx and anterior crop were already partially digested. This is in marked contrast to those species of *Conus* which feed on annelids. In these snails, digestion does not begin until the food is pushed from the posterior crop into the intestine.[5]

In order to photograph the feeding process in *C. striatus*, demersal fishes (the blennies *I. gibbifrons* and *E. marmoratus*) were used as prey organisms, since they

would remain more quiescent in the tank. In these cases the proboscis of *C. striatus* was extended along the surface of the substrate (Figs. 3 and 4) rather than vertically, as when a pelagic fish was the potential prey. The feeding process is shown in Figures 3–7 in photographs taken from a 16-mm. moving picture.

The venom of *C. striatus* did not affect the blennies as rapidly as it did *Kuhlia*. Blennies often struggled for a minute or more before breathing ceased and paralysis occurred.

A specimen of *E. marmoratus* 90 mm. in total length was easily consumed by a large (115-mm.) *C. striatus*. A smaller snail (shell length 81 mm.) easily consumed an *E. marmoratus* 85 mm. in total length. A specimen of *E. marmoratus* 100 mm. in total length struggled free, however, after being stung by the larger snail. Movements of the mouth and gills of this fish ceased within 2 minutes, and death followed within 5 minutes of stinging.

In the latter case the fish tore free of the dart, which remained held by the proboscis. In this instance as well as in several in which darts were ejected unsuccessfully, the proboscis, still holding the dart, was rapidly withdrawn into the mouth. Several minutes later the same dart was forced slowly out of the mouth, the process lasting 10–30 minutes. Each radula tooth, then, is used only once. If the prey is successfully captured, the tooth is swallowed with it; if not, the dart is later completely ejected and a new one is moved into place from the radula sheath.

Feeding in C. catus.—This is a smaller, intertidal species, often found on the reef flat in shallow pools isolated by the receding tide. Such pools are commonly frequented by the small blennnies and gobies on which this snail feeds.

In the laboratory, specimens were fed on the goby *B. fuscus*. The gastropods paralyzed and consumed with ease gobies equal to and somewhat greater than their own length (20–30 mm.). The feeding process is essentially identical with that described above for *C. striatus*.

Conclusion.—Two species of the gastropod genus *Conus* have been shown to prey on live fishes as their normal, and probably exclusive, source of food. This represents an ecological niche not previously known to be occupied by gastropods.

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ERRATA: STATISTICAL CORRELATION OF PROTEIN AND RIBONUCLEIC ACID COMPOSITION

In the article of the above title appearing in these PROCEEDINGS, **41,** 1011–1019, 1955, the following corrections should be made:

On page 1013, lines 10–11, for "... sixteen contain two (*aab*) ..." read "... twelve contain two (*aab*). ..."

On page 1015, legend to Figure 4, for "... same procedure as in Fig. 3 ..." read "... same procedure as in Fig. 2. ..."

GEORGE GAMOW AND MARTYNAS YČAS

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